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# NONASSOCIATIVE NUMBER THEORY

TREVOR EVANS, Emory University

**Introduction.** In several papers I. M. H. Etherington has studied the algebra of exponents of the general element in a nonassociative linear algebra and he has called these systems *logarithmetics*. If the linear algebra does not satisfy any identities the corresponding logarithmetic bears a close resemblance to the natural numbers. In fact, in [3] Etherington has shown that the elements of this particular logarithmetic can be defined in terms of partitioned classes in complete analogy to the Frege-Russell definition of the natural numbers as classes of classes. It is not too surprising then that we can also characterize this logarithmetic by a set of postulates analogous to the Peano postulates for the natural numbers. We do this in Section 1 and develop the basic properties of the logarithmetic in a manner paralleling the usual development of the natural numbers.*

We proceed to study the number theory of this logarithmetic. Several of the theorems in Section 1 and 2 including the "fundamental theorem of arithmetic" have been obtained by Etherington, although our derivations are in general quite different. Some of the standard theorems and conjectures of ordinary number theory have trivial analogues in this new number theory but a little more effort is needed to prove Fermat's Last Theorem.

By analogy with the extension of the natural numbers to the ring of positive and negative integers, we extend the logarithmetic to a system in which subtraction is always possible. The system so obtained is the left neoring of Bruck's recent paper [2]. The fundamental theorem of arithmetic has to be proved anew, since there are many more primes than just the original primes and their associates. In this new system we are also able to introduce the analogues of finite arithmetics and congruence by using some of the results of [6].

We conclude by mentioning a few problems and possible directions for further work.

**1. Peano-like postulates for the nonassociative natural numbers.** Peano's postulates characterize the natural numbers as a set closed under a unary operation and satisfying certain other conditions. If we replace the unary operation by a binary operation and make the corresponding changes in the postulates we obtain the following system.

*Undefined terms*: The set of nonassociative natural numbers, the elements of which we will just call numbers;† the binary operation of addition.

* See, for example, the first few pages in [8].

† We will always use the words "positive integer" in referring to the natural numbers of ordinary arithmetic.

*Postulates*:

(i) 1 is a number,

(ii) to every pair of numbers $a$, $b$ there corresponds a third called the sum of $a$ and $b$ and written $a+b$,

(iii) there are no numbers $a$, $b$ such that $a+b=1$,

(1) (iv) if the numbers $a$, $b$ and $c$, $d$ are such that $a+b=c+d$, then $a=c$ and $b=d$,

(v) if a set of numbers contains 1, and if whenever it contains numbers $a$, $b$ then it contains $a+b$, then the set contains all numbers. (The principle of nonassociative induction.)

Thus our numbers are $1, 1+1, 1+(1+1), (1+1)+1, 1+(1+(1+1)), \cdots$. By postulate (v), every number except 1 can be expressed as the sum of two other numbers and postulate (iv) implies that this can be done in only one way. Also, by postulate (iv), addition is in general noncommutative since $a+b=b+a$ implies $a=b$. Following Etherington, we will denote $1+1$ by 2, $1+(1+1)$ by 3, $1+(1+(1+1))$ by $4, \cdots$.

As an example of nonassociative induction we prove:

THEOREM 1. *For all numbers $a$, $b$, $a\neq a+b$.*

*Proof.* Let $S$ be the set of all values of $a$ such that $a\neq a+b$ for any $b$. $S$ contains 1 by postulate (iii). Let $m, n\in S$. If there exists a number $b$ such that $m+n=(m+n)+b$, then $m=m+n$ by postulate (iv), in contradiction to the assumption that $m\in S$. Thus $m+n\in S$ and so by the principle of nonassociative induction, $S$ contains all numbers.

An immediate consequence of this theorem is that there are no numbers $a$, $b$, $c$ such that $a+(b+c)=(a+b)+c$. That is, addition is completely nonassociative. Because of the lack of commutativity and associativity in addition, introducing an order or partial order into the system does not seem to be very fruitful. One fairly reasonable definition is as follows. We first define "well-formed part" of a number by (i) the only well-formed part of 1 is 1 itself, (ii) if $a=b+c$, the well-formed parts of $a$ are $a$ itself and the well-formed parts of $b$ and $c$. Now we define $x\leq y$ if $x$ occurs as a well-formed part of $y$, and $x<y$ if $x\leq y$ but $x\neq y$. With these definitions we get a partial ordering* between numbers but unfortunately $x<y$ does not imply $x+z<y+z$ or $z+x<z+y$. However, with the definition of multiplication given below, $x<y$ does imply $z\cdot x<z\cdot y$.

We introduce multiplication $a\cdot b$ (or $ab$) into our number system by

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* The partial order for $N$ can be extended to a complete ordering as was pointed out to me recently in conversation by R. H. Bruck and D. R. Hughes. Define $a<b$ in $N$ if (i) $|a|<|b|$; (ii) $|a|=|b|$ but $a_1<b_1$, where $a=a_1+a_2$, $b=b_1+b_2$; (iii) $|a|=|b|$ and $a_1=b_1$, but $a_2<b_2$. Unlike the partial ordering given above, this ordering has all the usual properties. Another complete ordering of $N$ is obtained by interchanging $a_1$ and $a_2$, $b_1$ and $b_2$ in (ii), (iii).

$$\text{(2)} \qquad \text{(i) } a\cdot 1 = a, \qquad \text{(ii) } a\cdot(b+c) = a\cdot b + a\cdot c.$$

Clearly this defines a product between every pair of numbers. We leave to the reader the proof of the next two theorems. Nonassociative induction is used on $a$ in the first and $c$ in the second.

THEOREM 2. *$1\cdot a = a$ for all numbers $a$.*

THEOREM 3. *$(ab)c = a(bc)$ for all numbers $a$, $b$, $c$.*

Some examples of calculation in our system are

$$2\cdot 2 = 2+2, \qquad 3\cdot 2 = 3+3 = (1+(1+1)) + (1+(1+1)),$$
$$2\cdot 3 = 2 + (2+2) = (1+1) + ((1+1) + (1+1)),$$
$$((3+2) + (3+2)) + (3+2) = ((1+2)+2)\cdot(2+1).$$

As an immediate consequence of the definition of multiplication the left-distributive law is satisfied. The cancellation properties of multiplication are given in the following theorems.

THEOREM 4. *If $xa = ya$, then $x = y$.*

*Proof.* This is true for $a = 1$. Assume that it is true for $a = m$ and $a = n$. Now, if $x(m+n) = y(m+n)$, expanding each side we get $xm + xn = ym + yn$. By postulate (iv), $xm = ym$, and so by our inductive hypothesis, $x = y$. Thus the theorem is true for $a = m+n$, and so for all values of $a$ by nonassociative induction.

In order to prove the other cancellation law it is useful to introduce the concept of length of a number $n$. We mean by this the positive integer obtained from $n$ by regarding $+$ in the expression for $n$ as the addition of ordinary arithmetic. We will denote the length of $n$ by $|n|$. The following relations hold.

$$\text{(3)} \qquad |m+n| = |m| + |n|, \qquad |m\cdot n| = |m|\cdot|n|,$$

*i.e.*, $m \to |m|$ is a homomorphism onto the positive integers.

THEOREM 5. *If $ax = ay$, then $x = y$.*

*Proof.* We use induction on $x$. When $x$ is 1, consideration of the lengths of the two sides of the equation $a = ay$ shows that $y = 1$. Consider the equation $a(m+n) = ay$. By the preceding sentence $y$ cannot be 1 and so $y = s+t$ for some numbers $s$, $t$. Then $a(m+n) = a(s+t)$ or $am + an = as + at$. By postulate (iv), $am = as$ and $an = at$.

Hence, if $am = as$ implies $m = s$, and $an = at$ implies $n = t$, then $a(m+n) = ay$ implies $m+n = y$. The theorem follows by nonassociative induction.

We now have a fairly complete picture of our nonassociative number system. Every number in it can be obtained from 1 by a finite number of nonassociative additions. Multiplication, $u(1)\cdot v(1)$, of two of these numbers satisfies $u(1)\cdot v(1) = v(u(1))$, in complete analogy with multiplication in ordinary arithmetic. Addition satisfies the uniqueness law, $a+b=c+d$ implies $a=c$ and $b=d$. Multiplication is associative, has 1 as an identity, is connected with addition by the left-distributive law, and satisfies the usual cancellation laws. In the language of modern algebra, this system can be described as additively the free groupoid generated by 1 with a multiplication introduced by $a\cdot b=b\phi_a$ where $\phi_a$ is the endomorphism of the groupoid determined by mapping 1 into $a$.

From now on, we will denote this system by $N$ and call it *nonassociative arithmetic*.

**2. Number theory.** We can now proceed with the development of the number theory of $N$. In view of the noncommutativity of multiplication we need the concepts of *left-factor* and *right-factor*. If $a=b\cdot c$, then $b$ is called a left-factor of $a$ and $c$ is called a right factor of $a$. If $b$, $c$ are not equal to 1 or $a$, we call them *proper* left- or right-factors. A number, other than 1, having no proper left-factors is called a *prime number*. Clearly, a prime number has no proper right-factors either. A striking property of factors in nonassociative arithmetic is given in the next theorem.

THEOREM 6. *If $p$ is a proper left-factor of $a$, and $a=b+c$, then $p$ is a left-factor of $b$ and a left-factor of $c$.*

*Proof*. Since $pq=a$ for some $q$ and $p\neq a$, then $q\neq 1$. Thus $q=m+n$ for some $m$, $n$. Then $p(m+n)=a$ and so $pm+pn=b+c$. By postulate (iv), $pm=b$, $pn=c$. That is, $p$ is a left-factor of both $b$ and $c$. We note that this theorem is not true if we consider right-factors instead of left-factors.

In ordinary arithmetic we have the theorem that if a prime is a factor of a product it is a factor of one of the numbers. The following theorem is similar.

THEOREM 7. *If the prime $p$ is a left-factor of the product $a\cdot b$, where $a$ is not* 1, *then it is a left-factor of $a$.*

*Proof*. We use nonassociative induction on $b$. For $b=1$ the theorem is certainly true. Now if it is true for $m$, $n$ and if $b=m+n$, then $p$ is a left-factor of $a(m+n)=am+an$. But $p\neq a(m+n)$ since $p$ is prime and so by Theorem 6, $p$ is a left-factor of $am$. Hence $p$ is a left-factor of $a$ by our inductive hypothesis.

The corresponding result for right factors is also true, but it is most easily obtained as a consequence of the following theorem.

THEOREM 8. (*The fundamental theorem of nonassociative arithmetic.*) *There is only one way in which a number can be written as a product of primes.*

*Proof*. Let $p_{|1|}p_{|2|}\cdots p_{|s|}$, $q_{|1|}q_{|2|}\cdots q_{|t|}$ be two products of primes such that $p_{|1|}p_{|2|}\cdots p_{|s|}=q_{|1|}q_{|2|}\cdots q_{|t|}$. By Theorem 7, $p_{|1|}$ is a left-factor of $q_{|1|}$ and since $q_{|1|}$ is prime, $p_{|1|}=q_{|1|}$. Then, by Theorem 5, $p_{|2|}\cdots p_{|s|}=q_{|2|}\cdots q_{|t|}$. Continuing this, we get $p_{|2|}=q_{|2|}$, $p_{|3|}=q_{|3|}$, $\cdots$. There must be the same number of factors in each product since otherwise we would eventually have 1 expressed as a product.

COROLLARY. *If the prime $p$ is a right factor of the product $a \cdot b$, where $b$ is not* 1, *then it is a right-factor of $b$.*

For $a \cdot b$ when written as a product of primes must end with $p$ by the theorem, and since this product of primes can be obtained by writing $a$ and $b$ separately as products of primes, $p$ must be the last factor in the expression of $b$ as a product of primes.

The concept of factor can be extended by defining $m$ to be a *factor* of $a$ if $a = smt$. The concept of mutually prime in ordinary arithmetic has several analogues in nonassociative arithmetic. Two numbers, $a$, $b$ are *mutually left-prime* if they have no common proper left-factor, *mutually right-prime* if they have no common proper right-factor, *mutually prime* if one is not a factor of the other, and no proper right-factor of one is a left-factor of the other. It is easy to verify the following generalizations of Theorems 6, 7:

(i) If $m$ is a proper factor of $a$, not a right-factor, and $a = b + c$, then $m$ is a factor of $b$ and of $c$, (ii) let $m$ be a factor of $a \cdot b$; if $m$, $b$ are mutually prime, then $m$ is a factor of $a$, or if $a$, $m$ are mutually prime, then $m$ is a factor of $b$.

If $a$, $b$ are two mutually left-prime numbers then $a$, $a+b$, $(a+b)+b$, $((a+b)+b)+b, \cdots$ have no nontrivial left-factors by Theorem 6, and so are all prime. Hence there are an infinite number of primes. An example of such an infinite sequence of primes is 2, 3, 4, $\cdots$. The twin primes conjecture of ordinary arithmetic has a trivial generalization for, if $k$ is any number, there exists an infinite number of pairs of primes of the form $n$, $n+k$. The analogue of Goldbach's conjecture fails to hold by virtue of postulate (iv). However, another famous conjecture of ordinary arithmetic is provable in nonassociative arithmetic. In fact, an even stronger result than the original is true.

THEOREM 9. (*Fermat's Last Theorem*). *There are no numbers $x$, $y$, $z$ such that $x^{|n|} + y^{|n|} = z^{|n|}$ for any positive integral $|n|$ greater than $|1|$.*

*Proof.* We obtain a proof by contradiction. Let $x$, $y$, $z$ be numbers such that $x^{|n|} + y^{|n|} = z^{|n|}$, where $|n|$ is a positive integer greater than $|1|$. We note that (i) neither $x$ nor $y$ can be 1 since this would imply that $x^{|n|} + y^{|n|}$ is a prime, (ii) $|x|^{|n|} + |y|^{|n|} = |z|^{|n|}$.

Since $|n|$ is greater than $|1|$, $z^{|n|}$ has $z$ as a left-factor and so by Theorem 6, both $x^{|n|}$ and $y^{|n|}$ have $z$ as a left-factor. Let $x$ and $z$ be expressed as a product of primes in the form $x = p_{|1|}p_{|2|} \cdots p_{|s|}$, $z = q_{|1|}q_{|2|} \cdots q_{|t|}$. Then, since $zu = x^{|n|}$ for some number $u$, we have $q_{|1|}q_{|2|} \cdots q_{|t|}u = p_{|1|}p_{|2|} \cdots p_{|s|}v$, where $v = x^{|n-1|}$.

By Theorem 7, $q_{|1|} = p_{|1|}$, $q_{|2|} = p_{|2|}, \cdots$. Now $|s| < |t|$, for otherwise $|z| \leq |x|$, in contradiction to $|x|^{|n|} + |y|^{|n|} = |z|^{|n|}$. Hence, $z = xa$ for some $a$ and, similarly, $z = yb$ for some $b$.

We have then $|z| = |x| \cdot |a|$, $|z| = |y| \cdot |b|$. Substituting in $|x|^{|n|} + |y|^{|n|} = |z|^{|n|}$ for $|x|$ and $|y|$, we get $|1|/|a|^{|n|} + |1|/|b|^{|n|} = 1$. This is a contradiction since for $|n| > |1|$, no positive integers $|a|$, $|b|$ satisfy such a condition.

**3. Introduction of "negative integers."** In ordinary arithmetic, zero and the negative integers are introduced in order that subtraction always be possible. The same problem arises naturally in nonassociative arithmetic also. In $N$, subtraction can be defined between some pairs of numbers as follows. If $m=p+n$ we can introduce the operation of right-subtraction $m-n$ between $m$ and $n$ and write $m-n=p$. If $m=n+q$, we can introduce the operation of left-subtraction* $-n+m$ between $m$ and $n$ and write $-n+m=q$. With these definitions we get the following properties

$$(4)\qquad \begin{aligned} (m+n)-n&=m, & n+(-n+m)&=m,\\ (m-n)+n&=m, & -n+(n+m)&=m. \end{aligned}$$

Clearly these are properties we would like subtraction to have, and in a general nonassociative system they are the most for which we can hope. The problem now is to find a system containing $N$ and such that left- and right-subtraction is possible between every pair of elements. More specifically, we want a system with two operations $+$, $\cdot$, and such that (i) the equations $a+x=b$, $y+a=b$ have unique solutions, (ii) multiplication is associative, (iii) the multiplicative identity 1 generates the system, (iv) the cancellation laws hold, (v) the left-distributive law holds, (vi) with respect to the operation $+$, 1 generates a subsystem isomorphic to $N$. Such a system is of the type discussed by Bruck in a recent paper [2] and called by him a left neoring. However, there are many left neorings satisfying the above conditions. We will choose the one which seems to be the most natural extension of $N$.

Let $L$ be the free monogenic loop† generated by 1 with the operation written as addition. This is the nonassociative analogue of the additive group of integers. The mapping $1\rightarrow a$ where $a$ is any element of $L$ determines an endomorphism $\phi_a$ of $L$ and we can introduce a multiplication into $L$ by defining $a\cdot b=b\phi_a$. We will denote the resulting system by $I$ and call it the left neoring of nonassociative integers. In this section "number" will refer to an element of $I$.

An immediate consequence of this definition of multiplication is that $a(b+c)=ab+ac$ for all $a$, $b$, $c$. In addition, as is shown in [2], multiplication is associative and the two cancellation laws of multiplication are satisfied. Since, additively, $I$ is a loop, we do have the required subtraction properties, and the subsystem of $I$ consisting of $1$, $1+1$, $1+(1+1)$, $\cdots$, *etc.* is isomorphic to $N$. We refer the reader to [2], [6], for a discussion of the algebraic structure of $I$. We wish to introduce here some analogues of ordinary number theory in $I$. For this reason we will use another approach to the system which has the advantage of an explicit representation of its elements.

Consider all expressions which can be generated by 0 and 1 with the three

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* Note that the $-$ and $+$ here do not exist independently, but are each part of the notation for the binary operation of left-subtraction.

† For a discussion of free loops see [1], [4], [5].

binary operations of addition $a+b$, left subtraction $-a+b$, and right subtraction $a-b$. We call such expressions *numerical expressions*. An example is $(4+(0-1))-((1+1)+(1+(-2+1)))$, where 2, 4 have the usual meaning as abbreviations.

Two numerical expressions are equal if and only if their equality follows from the following

$$
(5)\qquad
\begin{array}{rll}
\text{(i)} & a+0=0+a=a, & \\
\text{(ii)} & a-a=-a+a=0, & \\
\text{(iii)} & a-0=-0+a=a, & \\
\text{(iv)} & (a+b)-b=a, & -b+(b+a)=a, \\
\text{(v)} & (a-b)+b=a, & b+(-b+a)=a, \\
\text{(vi)} & a-(-b+a)=b, & -(a-b)+a=b,
\end{array}
$$

where $a$, $b$ are numerical expressions.

Clearly (i), (ii), (iii) are properties we wish 0 to have, (iv) and (v) are the properties of subtraction we already have in $N$. Equations (vi) are actually consequences of the preceding equations and we list them merely for their usefulness in computation. We remark that $(-a+b)$ is the unique solution of $a+x=b$ and $b-a$ is the unique solution of $y+a=b$.

Our nonassociative integers are now defined as the classes of equal numerical expressions. A multiplication is introduced into the system by $u(1)\cdot v(1)=v(u(1))$ where $u$, $v$ are numerical expressions.

That this system is $I$ is a consequence of the results of [4], [5]. Another result from [4, Theorem 2.2], shows that in each class of equal numerical expressions there is a unique expression of shortest length (here "length" refers to the number of 0's and 1's in the expression). Such a shortest numerical expression is characterized by the property that there is no application of equations (5) to the expression which will shorten it. We will call this the normal form of the class of equal numerical expressions and refer the reader to [4], [5] for a full discussion of these ideas.

The following examples illustrate the rules of computation in $I$ and some specific computations.

$$
(6)\qquad
\begin{array}{rl}
\left.\begin{array}{rl}
\text{(i)} & a\cdot 1=1\cdot a=a, \\
\text{(ii)} & a\cdot(m+n)=a\cdot m+a\cdot n, \\
\text{(iii)} & a\cdot(m-n)=a\cdot m-a\cdot n, \\
\text{(iv)} & a\cdot(-m+n)=-a\cdot m+a\cdot n,
\end{array}\right\} & \text{by the definition of multiplication,} \\
\text{(v)}\ \ a\cdot 0=a(1-1)=a-a=0, & \\
\text{(vi)}\ \ 0\cdot a=0, \text{ by induction on the length of } a, & \\
\text{(vii)}\ \ a\cdot(0-1)=0-a, \quad a\cdot(-1+0)=-a+0, &
\end{array}
$$

$$\text{(viii)}\quad (0-1)\cdot(-1+0) = -(0-1) + (0-1)\cdot 0 = -(0-1) + 0 = 1,$$

$$\begin{aligned}\text{(ix)}\quad (1-2)\cdot((0-1)+2) &= (1-2)\cdot(0-1) + (1-2)\cdot(1+1)\\ &= (0-(1-2)) + ((1-2)+(1-2)).\end{aligned}$$

The discussion of the number theory of $I$ is complicated by the existence of units. As usual we define a unit to be an element possessing a multiplicative inverse. In the ring of integers of ordinary arithmetic there are only two units but the left neoring $I$ contains an infinite number.

We will call the elements $0-1$, $0-(0-1)$, $0-(0-(0-1))$, $\cdots$ the first, second, third, $\cdots$ right negatives of 1 and similarly, $-1+0$, $-(-1+0)+0$, $-(-(-1+0)+0$, $\cdots$ the first, second, third, $\cdots$ left negatives of 1. It is easily verified from equations (5) and (6) that the product of the $n$th left negative and $n$th right negative is 1.

It is not quite so easy to show that these are the only units in $I$. We recall that the product of two elements $u(1)$, $v(1)$ of $I$ is defined by $u(1)\cdot v(1)=v(u(1))$. Hence we have to show that the left- and right-negatives of 1 are the only elements of $I$ which satisfy $v(u(1))=1$. This is an immediate consequence of Lemma 2 in [**5**].

Since $(0-1)^2=0-(0-1)$, $(0-1)^3=0-(0-(0-1))$, $\cdots$ and $(-1+0)^2=-(-1+0)+0$, $(-1+0)^3=-(-(-1+0)+0)+0$, $\cdots$, the units of $I$ are exactly the powers of $0-1$.

We collect these results as a theorem.

THEOREM 10. *The multiplicative group of units of $I$ is the infinite cyclic group generated by* $0-1$.

As before $b$ will be called a *left-factor* of $a$ if there exists an element $c$ of $I$ such that $b\cdot c=a$. If neither $b$ nor $c$ is a unit and $a\neq 0$, we say that $b$ is a *proper left-factor* of $a$. In the same way we define *right-factor* and *proper right-factor*. We note that any number is a factor of 0. Two numbers $a$, $b$ in $I$ will be called *associates* if $xay=b$ where $x$, $y$ are units.

LEMMA 1. *If $a$, $b$ are left- (right-) factors of each other, then they are associates.*

*Proof.* If $ax=b$, $by=a$, then $axy=a$ or $xy=1$. Hence $x$, $y$ are units. The proof for right-factors is similar.

If $a$ is a left- (right-) factor of $b$ and $b$ is a right- (left-) factor of $a$, then $a=b$ unless both $a$ and $b$ are units. A proof of this leans heavily on the results of [**4**], [**5**], and so we omit it.

In ordinary arithmetic, the primes in the ring of integers are simply the original primes in the set of natural numbers multiplied by the units. This situation does not carry over to nonassociative arithmetic. In fact, a rather complicated situation exists in $I$. We define, in the usual way, a *prime number* of $I$ to be a number without proper factors. Then all the primes of $N$ are primes in $I$.

We also have primes such as $0-(1+1)$ consisting of the product of the prime $(1+1)$ in $N$ and the unit $0-1$. But other primes such as $1-(1+1)$ exist in $I$, not the product of a unit and a prime of $N$. In addition, there is a special subclass of the primes of $I$ with the property that no prime in this subclass can be written as a product of two numbers of shorter length. For want of a better name, we will call these *special primes*. The number $(0-(1+1))$ is a prime but it is not a special prime since $0-(1+1)=(1+1)\cdot(0-1)$. However, $(1+1)$ is a special prime and, more generally, all the primes of $N$ are special primes in $I$. Examples of other special primes are $(1-2)$, $(1-3)$, $(1-4)$, $\cdots$.

We now state some theorems, giving only brief outlines of the proofs, which are basic in the further development of the number theory of $I$.

THEOREM 11. *Let $a$ be an element of $I$ represented by a numerical expression in normal form so that $a$ has one of the forms $m+n$, $m-n$, $-m+n$ where $m$, $n$ are numerical expressions. Then any proper left-factor of $a$ is a left-factor of $m$ and $n$.*

*Proof.* This corresponds to Theorem 6 for $N$. The proof proceeds by ordinary induction on the length of $a$, coupled with the fact that the representation of a number as a numerical expression in normal form is unique.

THEOREM 12. *If the prime $p$ is a left-factor of the product $ab$, where $a\neq 1$, $b\neq 0$, then $p$ is a left-factor of $a$.*

*Proof.* By induction on the length of $b$, and by the previous theorem.

THEOREM 13. *If a number $a$ can be written as a product of primes in two ways, say, $a=p_{|1|}\cdots p_{|s|}$ and $a=q_{|1|}\cdots q_{|t|}$, then $|s|=|t|$ and $p_{|i|}$, $q_{|i|}$ are associates ($|i|=|1|,\cdots,|s|$).*

*Proof.* By Lemma 1, Theorem 12, and the left-cancellation law for $I$.

We conclude our discussion of $I$ by introducing the concept of congruence in it. In ordinary arithmetic, a homomorphic image of the ring of integers is obtained by adding the relation $|m|=|0|$ to the ring. Then two integers are congruent mod $|m|$ if they map onto the same element under this homomorphism. It is shown in [6] that if $m(=u(1))$ is an element of $I$, then adding the relation $u(1)=0$ to $I$ determines a left neoring which is a homomorphic image of $I$. We define two numbers in $I$ to be congruent mod $m$ if they map onto the same element under this homomorphism. Alternatively, we can define two numbers in $I$ to be congruent mod $m$ if their difference lies in the fully invariant normal subloop, generated by $m$, of the additive loop of $I$. The relation between these two points of view is discussed briefly in [6] and can be studied in detail using the techniques of [4], [5]. The homomorphic images of $I$ described above are the nonassociative analogues of finite arithmetics.

With the above definition of congruence in $I$, some of the elementary properties of congruence in ordinary arithmetic carry over without difficulty (see, *e.g.*, Chapter 1 in [8]). The author does not know whether the same is true of some of the deeper theorems involving congruence.

**4. Further developments.** The ideas introduced in this paper can be developed in several directions. There are many problems for the arithmetic $N$, *e.g.*, obtaining an analogue of the prime number theorem. This seems quite feasible since estimates of the number of nonassociative natural numbers of given length are available.

In some of our proofs of properties of $N$ we used properties of ordinary arithmetic including induction. Can this be avoided completely and all properties of $N$ obtained from the postulates for $N$ given in Section 3? One way to do this is to develop ordinary arithmetic within $N$. Define nonassociative powers of numbers in $N$ by $a^1 = a$, $a^{m+n} = a^m \cdot a^n$ where $m, n \in N$. An equivalence relation, $\equiv$, between numbers in $N$ can be defined by $m \equiv n$ if $a^m = a^n$ for all $a \in N$. We now show that the set of these equivalence classes satisfies Peano's postulates for the ordinary natural numbers. This is a consequence of the Peano-like postulates which $N$ satisfies. The length of a number $n$ in $N$ is defined as the equivalence class containing $n$. In this way all of ordinary arithmetic and in particular those parts which we have used in discussing $N$ can be developed inside $N$. It follows that if we set up $N$ as a formal system, there will be a Gödel incompleteness theorem for the system. We leave to the interested reader the detailed carrying out of the above ideas. A related topic which may be interesting is the theory of recursive functions of nonassociative natural numbers.

There are many concepts involving congruence in ordinary arithmetic which should have interesting analogues in the arithmetic $I$. In particular we can ask such questions as the following. For what congruences does the quotient arithmetic (i) satisfy the cancellation law, (ii) allow division, (iii) allow unique division, (iv) satisfy the commutative laws of addition and multiplication? Other problems are (i) what is the structure of the multiplicative semigroup of $I$, (ii) can $I$ be embedded in a system with division?

If we add to $N$ the identical relation $(a+b)+(c+d)=(a+c)+(b+d)$ we get an arithmetic $S$ with many interesting properties. In this system, which is the free symmetric groupoid generated by 1 (see [7]), we define multiplication as usual by $u(1)\cdot v(1) = v(u(1))$. Then multiplication is commutative and so both distributive laws hold. This arithmetic is extremely close to ordinary arithmetic, differing only in the replacing of the associative and commutative laws by the single law $(a+b)+(c+d)=(a+c)+(b+d)$. A study of the number theory of $S$ should lead to some interesting problems. Another problem which presents itself is the obtaining of a set of Peano-like postulates which characterize $S$.

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# MAINTAINING COMMUNICATION*

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Some twenty years ago a professor of philosophy spoke to the mathematics club at the University of Virginia. The speech was followed by a warm discussion of the paradoxes of Zeno. Some regarded the mathematical explanation of the paradoxes as completely adequate, others disagreed, and needless to say, each disputant emerged triumphantly bearing the opinion he had carried in. But one remark of a professor of physics has stayed with me ever since. He said that he could easily conceive that someone could arrive in his own mind at a perfect solution of the paradoxes and still be unable to convey the explanation to anyone else.

Let us at least temporarily suspend disbelief in this philosopher with the incommunicable thoughts, and not boggle over reasons for acceding to him a belief that we deny to Fermat and his celebrated proof that was too long for the book's margin. There remains the fact that to the body of philosophy he remains exactly as useless as though he had never existed. Perhaps he has derived intense personal satisfaction from his brilliant reasoning, but the rest of the world may as well ignore him.

In recent years I have been troubled by a suspicion that this image of the uncommunicative philosopher may be a parable of an approaching state of mathematics. Fortunately, rather than a parable it is an overdrawn caricature, but as in all caricatures some features are recognizable. For in mathematics, as in the sciences, the communication of ideas becomes steadily more difficult. This is a matter that concerns all of us, and each of us should try to help in keeping the lines of communication open.

To begin with, there are the mechanical and financial difficulties involved

* Retiring Presidential Address, delivered at the Fortieth Annual Meeting, December 29, 1956, Rochester, N. Y.

in the publication of a flow of mathematical research whose volume grows steadily. I do not intend to say much about this. Competent people are trying hard to find a solution, and it is not entirely clear that a solution exists. As someone said to me earlier this month, we have caught a geometric progression by the tail. Nevertheless, as matters now stand the author of a reasonably good paper can expect to have it published with reasonable speed. I propose to talk about the author and the paper, rather than about its publication.

I wonder if any one here has already silently objected that I seem to be speaking of research, and not of teaching, which is a more proper activity of the Association. If so, his objection is itself an example of a hindrance to communication. The normal activities of a mathematician should include a mixture of three ingredients. One is teaching; another is research; and the third, which as Professor Duren says is all too often left unmentioned, is scholarship. Any one of these three may be omitted, but in my opinion not without loss. No one can confine himself to any single one of the three without injury to himself, and perhaps to others too. When this Association was formed in 1915 it was for the satisfaction of a need. To put it bluntly, the American Mathematical Society had refused to consider anything but mathematical research as being in its sphere of activity. Nevertheless, I consider it regrettable that the need existed, and that the activities of teaching and research are thus separated. One of the visible results of the separation is seen in the content of mathematics courses in colleges, and even more in high school courses. The tremendous changes in matter and method of research have produced hardly a ripple at the freshman level, and less in the high schools.

Let me enlarge a bit on this topic. During our lives mathematicians have transferred their chief interest to new topics, and they study these topics with changed thought-patterns. Abstraction and generalization have increased, not (at least not always) merely for the sake of generality itself, but because the abstract approach exposes the essential ideas and clears away what is merely fortuitous, and the generality yields new fields of application of the ideas. Set-theoretic concepts have pervaded mathematics, both pure and applied. The basic ideas of topology are the common language of mathematicians in all fields. Linear processes and matrix algebra are now matters of great interest to applied mathematicians, and so on, through a list which is quite long if we include more specific items. Yet there are modern books on "mathematics for engineers" which explain, as did my grandfather's calculus text, that there are two kinds of zero, the "nothing-at-all" kind obtained by subtracting a number from itself, and another mysterious kind, smaller than any positive number but yet not reconciled to being nothing at all, that we meet when we "evaluate the indeterminate expression 0/0." Most mysteriously, this nonsense is written in a presumably sincere effort to help the reader understand the subject. Fortunately, there now exists a respectable minority of calculus texts in which definitions and proofs are carefully and correctly stated. At a lower level, the advent first of the electric computing machine and then of the high-speed electronic computer has diminished the importance of some older computational devices. Horner's method can profitably be forgotten; yet in some good high schools several weeks are allotted to it. Logarithms are not nearly as important as they once were, and the various "time-saving" devices for solving triangles by formulas adapted to computation with logarithms are hardly worth the time they cost. Yet, if I hear a chorus of mathematicians' voices (and I hear my own voice in the chorus) asking "Why do these people spend irreplaceable time on the less valuable subjects and ignore the better ones?," I also hear voices (including the voice of my conscience) asking, "Did you bother to tell them?."

Here, at least, I can add a more cheerful note. The Association's Committee on the Undergraduate Program is composed of mathematicians who are active participants in research and are also interested in undergraduate teaching, and its aim is to provide the means of changing the present undergraduate teaching by making some of the more recent gains in mathematics available to college students. Also, there are other committees and organizations working on related problems, so we may reasonably hope for improvement.

Communication between the teacher and the scholar of mathematics is so important that I can imagine only one satisfactory arrangement: the two must wear the same skin. The man who is neither researcher nor scholar, and nevertheless is listed on some payroll as teacher of mathematics, is probably falsely listed. At best he is a transmitter of information given him in the more or less remote past; at worst he is drill-master for the problems in some third-rate textbook. In either case, he hasn't bothered to come to this meeting. The man who is a research specialist and teacher, without being a scholar, is easy enough to find. Sometimes he is a teacher only by necessity, giving all his enthusiasm to his research. Sometimes he is an enthusiastic teacher of advanced students, spreading forth the pleasures of specialization in his own field. In this case he may produce students with great enthusiasm but narrow views, whose broader education must be left to other more scholarly teachers or the hands of the gods. In either case, he gives weight to the definition in *Webster's Collegiate Dictionary*:

> **doctor** [O.F. *doctour*, fr. L. *doctor* teacher, fr. *docere* to teach] **1.** *Archaic.* A teacher; a learned man.

Quite another matter is the communication of thought between the research worker on the one hand and the teacher-scholar on the other. Now we are in a zone where the individual mathematicians can begin to feel their own share of guilt in a situation that is far from perfect, although a prevailing climate of thought is also largely to blame. Here is the domain of usefulness of the expository article. The advanced treatise or monograph may be serviceable as a means of showing all that has been accomplished in some field, but for the man who is principally a teacher or principally a researcher in some other field, the sheer bulk of the monograph may be discouraging. Besides, these are properly the fruits of long labor, and their writing is not lightly to be undertaken. But for expository writing on a smaller scale there is considerably more demand than supply. In order to have at least one number to quote, to substantiate my claim that the supply is scant, I thumbed through the first hundred pages of the latest number of *Mathematical Reviews*, searching for papers dismissed with the comment "Expository paper." I found seven of them, that is roughly one expository article to a hundred research articles. But more careful investigation shows that two of the seven were reports of conferences, hence hardly to be considered expository in the present sense. Of the remaining five, four were in Russian. Are we then to dismiss the scholar, and the research worker in some other field, and the physicist or chemist or interested layman, with the advice "Learn Russian and subscribe to the *Uspehi*"?

Certainly this trouble has been recognized before, and various efforts made to cure it. For example, the Carus Monographs and the Slaught Papers are sponsored by this Association, and some of the "What is . . . ?" series in the MONTHLY were excellent. But it has always been difficult to keep up the supply.

A chief reason for this scantiness is easy to find. Every normal human being wants recognition for his work. Even the hypothetical philosopher with his incommunicable resolution of Zeno's paradoxes would probably like to hear a word of praise from some believing soul. Since a research paper, even on the narrowest and most special of topics, is ordinarily looked on with more reverence than even an excellent expository paper, it is natural and human that a mathematician should be inclined to spend all his available working time on research. This is particularly true of the younger mathematicians. With them, recognition and promotion may well depend more on published research than on scholarship or teaching. Besides this, the writing of a good expository article calls for breadth of view and historical perspective, which are hardly the correlates of youth.

But these reasons apply much less cogently to the mature mathematician, who has reached a secure position and a level of esteem not likely to be greatly raised or lowered. By "mature" I do not mean superannuated. I am not recommending the writing of expository papers as a sort of pastime for gentlemen (young, old or middle-aged) who have determined by careful self-examination that they haven't a research paper left in their systems. A man of thirty may have attained position and recognition and broad knowledge; a man past seventy may be active in research, as the current volume of the *Proceedings of the National Academy of Sciences* will show.

The fund of scholarship that supports the writing of a good expository paper comes from having read, marked, learned, and inwardly digested many articles in some field. Usually this reading was prompted by an interest in some research problem, but this is not a necessary condition. However, the mere intensive reading of many papers is not a sufficient condition either. One can be so imbued with interest in one special problem that everything is mapped on a sort of polar coordinate system, with the one special problem at the origin and interest inversely proportional to $r^2$. Alternatively, one can read something new to find what it is in itself and how it relates to previous knowledge. Such reading, with thoughtful rumination, is a natural source of scholarship, of good teaching, and in particular of good expository writing.

Leonard Eugene Dickson used to say (I have forgotten his exact words) that every mathematician owed a debt to mathematics that he should repay by one hard job of scholarly writing. His was the huge *History of the Theory of Numbers*. Not many of us could consider such a vast undertaking. But each of us owes the debt, and should not repudiate it if he is mathematically solvent.

The purpose of an expository mathematical paper is, of course, to convey information about some domain of mathematics to some audience, which might consist of specialists in a slightly different field who wish to add to their research capacities, or of mathematicians wishing to broaden their knowledge, or of teachers at any level from high school to graduate school. Obviously, the best written exposition fails if no one reads it. If any man wishes to consider himself a teacher and scholar of mathematics, it is his clear duty, and it should be his pleasant duty, to add continually to his knowledge, and in this he should be greatly helped by expository articles of the type appropriate for him. The teacher who neglects this is doing himself a grave wrong. What is worse is that he is doing an even graver wrong to his students and to his subject. I have heard of a professor (not a mathematician) who remarked that he had been lecturing twenty years from the same notes, and that the only change that he expected after another twenty years was that the pages would be yellower. I cannot conceive of any field in which this attitude would be harmless. But certainly any teacher of mathematics who would thus decide that mathematics belongs in the Valley of Dry Bones must inevitably convey the same impression to his students. The best of books is inferior to a human being as a means of conveying enthusiasm for and pleasure in a field of study, and if the teacher fails to show that the subject is alive and moving and fascinating, he fails in just that respect in which the responsibility is most peculiarly his own.

One form of communication which up to the Second World War had been in rather bad shape, but fortunately is now improving, is the contact between mathematicians and other scientists. Applied mathematics was not long ago regarded as the tedious solution of specific problems by known devices, and often without adequate logical justification. Even today the mathematical reasoning in the quantum theory of fields or in nuclear physics is apt to shock a mathematician trained in rigor. There are two ways of reacting to such a situation. One is to look on the physical theory as ludicrous and refuse to sully one's hands with it. The other is to observe that the illogical theory has yielded useful results, and is thus probably a sort of parody of a rigorous and coherent theory. This implies a challenge to find that rigorous theory. It is in such situations as this that the applications have benefited mathematics, by calling for new devices and new combinations of old devices to handle a problem which has presented itself not artificially but irrepressibly and clothed with its own importance.

Mathematics has retained its place among the chief subjects of education for well over two thousand years, but this does not mean that it is surely immortal. It has stood firmly because it has stood on two legs. First, it is supported by its innate beauty and austere elegance. Second, it is supported by its usefulness to scientists and technicians of all kinds. If we try to make it stand exclusively on its usefulness, it becomes a mere tool for the use of non-mathematicians, and degenerates into dullness and eventually into uselessness. If we try to make it stand exclusively on its esthetic virtues, we not only make it useless to other sciences but reject the stimulus that it can receive from them. So it is desirable that among the research workers in mathematics there should always be some who are interested in its applications. Likewise, the scholars and the teachers should not ignore the many uses of mathematics. Right now this places quite a demand on the scholars and teachers, for mathematics has entered new fields, sometimes in rather unexpected ways. As examples, I cite genetics and the theory of games.

So far, all the aspects of communication that I have discussed have borne the typical earmarks of a Mathematical Association activity—they have all called for much individual enthusiasm and activity and very little cash. I now wish to say something about another activity which now involves millions of dollars, but at its outset had the typical Association earmarks. I am referring to the Institutes for teachers. A while ago I made the obvious remark that enthusiasm for a subject is easiest conveyed by personal contact with an enthusiastic teacher. It is reasonable to assume that a teacher of mathematics brings some store of enthusiasm to his teaching. But we ask a great deal if we expect that teacher to keep up his enthusiasm and increase his scholarship by reading and study if he himself has no contact with some other enthusiast. Clearly it is desirable to rekindle the enthusiasm and increase the learning of teachers by giving them the chance to study occasionally with leading mathematicians. This is the motive of the Institutes for teachers. These have sprung up in various forms; in time-scale, they varied from short conferences up to the one developed at Notre Dame, which is devised especially for teachers and has a program based on five summers' attendance and leading to the degree of M.S. The one about which I now want to speak took place in the summer of 1953, at the University of Colorado. The enthusiasm and personal dedication were certainly there; Burton W. Jones did two men's work. Other mathematicians were vigorous in their cooperation, and the University of Colorado supported the project well. The supply of funds was none too ample, but the personal contributions made the Institute a notable success. Its chief disadvantage, of course, was that it alone could not reach a vast number of teachers.

Only three years after the Colorado Institute, Congress, alarmed by our shortage of scientists and technicians, made an appropriation of millions of dollars for improvement in teaching of mathematics and science. The National Science Foundation had to find wise ways of using this money for the given purpose. It must have been quite a help to them (I speak from no expert knowledge) that they did not have either to spend the time needed for a trial run or to take the risks of putting the money into new and untried devices. Summer Institutes had already been tried out and found successful, and the experience was there for guidance. Now from small beginnings we have progressed to a large enterprise. The scale is utterly beyond the Association's financial resources, but the personal contribution is still a vital need. Some of us may be needed as teachers in the institutes; others may wish to attend them for refreshing. I hope that each of us will do whatever he can to help with the project. It would be a serious error to think that the large scale of operation has diminished the importance of the individual contributor. Quite the opposite is true. The success of an Institute depends above all on the presence of workers who have both learning and enthusiasm, and the increase in the number of institutes calls for the cooperation of every one who can make a contribution.

We have been hearing a great deal recently about the proliferation of mathematical research. This certainly is no news to anyone who has noticed the steady growth in the size of *Mathematical Reviews*. About sixteen years ago I bought a house which had been unoccupied for five years. Summer after summer we fought a losing battle against honeysuckle; it grew faster than we could dig it out. Then came the discovery of the weed-killing properties of 2-4-D, and the battle was won. As I understand it, a broad-leaved plant sprayed with 2-4-D does not die at once; it begins to proliferate rapidly, growing quickly and without organization. As a result, it dies. I cannot look on the proliferation of mathematics as being in all circumstances an unqualified good. Each mathematical discipline needs to draw on the others; yet it is impossible for even the best of us to keep abreast of the research in more than a small part of the field. We are separating off into small groups of specialists with little intercommunication.

What the solution is, or whether there is a solution, I do not pretend to know. But I can propose a palliative. We can pay more attention to the quality of our writing. Within a generation there has been a regrettable decline in the style and clarity of mathematical composition, a decline visible in all countries. Even the French have slipped sadly from their previous excellence. This decline has made it harder for us to read articles in any field in which we are not specialists. A published paper is not properly an open letter to a half-dozen fellow-specialists who understand the motivation, possess the antecedent information and can readily fill in omitted definitions and proofs. An addition to mathematical knowledge has significance as a part of the body of all mathematical knowledge, and its place in that body should be clearly indicated. Unless the reader is one of the tiny group of those who have read everything in that specific subject, he may be quite capable of following the details of proofs and yet unable to realize why the author was led to study the problem, or how it complements past knowledge or points toward future advances. If the author has not bothered to establish the setting of his paper and convey some of the motive that impelled him, he has driven off potential readers and made everything harder for those who stay with him. I am convinced that it is the duty of editors to demand, not to forbid, the writing of introductory paragraphs to provide motivation and background. This need not be long; even a little can be disproportionately helpful.

Within the body of the paper, it is often distressing that so little care is expended on presentation. Often the author has clearly worked hard on the mathematical content, removing all superfluous hypothesis, obtaining as much conclusion as possible without irrelevant steps. But after he has burnished the mathematics to a high polish, he has written it down hurriedly, with little thought for style. Long sentences drag their gangling dependent clauses across the page. Pronouns look back helplessly into a welter of nouns in the hope of finding an antecedent. Even worse for the uninitiated reader, there are expressions that proclaim the hopelessness of the search. Many papers bristle with "The expression . . . is defined in the obvious way," and "Clearly, . . . ," and "By the usual proof, . . . ," and "Mapping this onto $A$ in the natural way," and "By a well-known theorem, . . . ." These are frequently not even space-savers; the uninformative words can be replaced by a precise statement, and the "well-known" theorem precisely located, with little if any cost in length and with great benefit to the intelligent but imperfectly informed reader.

Moving still further in this direction, we meet a mental attitude that regards communication as vulgar. There are mathematicians (fortunately few) who consider that a speaker has somehow "lost face" if he has spoken so as to be intelligible to any but the select few. One hears of "folk theorems," established (presumably) by some expert, communicated verbally or more likely mentioned in an off-hand way during some conversation with another expert or two, and thereafter unpublishable forevermore because no one would want to publish a "known theorem." This is not mere uncommunicativeness; it is active opposition to communication. I recently saw it nicely summarized in a sentence in one of the reviews in a recent number of *Mathematical Reviews*; I am quoting from memory because I have forgotten the name of the reviewer and prefer to forget just who said this: "This theorem seems not to have been published previously, but is probably known to some of the workers in this field."

I would much prefer to see such mathematicians converted to fellow-workers. However, if they wish to form a Society for Mutual Admiration, or a sort of Egyptian priesthood guarding their secret knowledge from profanation by the vulgar, we can only let them go their way. But it would be surely wrong to abet them in their self-satisfaction. If they choose to suppress their knowledge, they cannot ask for recognition because they possess it. No man deserves serious credit for having established a theorem until he has put the proof in plain sight, for the open criticism of other mathematicians. I know of at least two instances in which a report was widely circulated that Mr. X had a proof of this or that, and later it turned out that what he had was seriously defective. This is quite excusable; men do make mistakes. But it would have been inexcusable to ask for the acclaim of mathematicians on the strength of a "proof" with an error in it; and it is likewise wrong for us to give acclaim on the strength of a proof that has not been openly presented for criticism and found sound. I feel that no mathematician should hesitate to publish a result because someone tells him that he thinks that Z did something like that two years ago but didn't write it up; nor should he falsify history by giving credit to Z for priority when Z has not established his right to the credit by letting us see his proof.

Everyone of us is touched in some way or other by the problems of mathematical communication. Every one of us can make some contribution, great or small, within his own proper sphere of activity. And every contribution is needed if mathematics is to grow healthily and usefully and beautifully.

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# A CURIOUS SEQUENCE OF SIGNS

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In this paper we give an algebraic identity which gives rise to a sequence of number theoretic functions, $u_b(n)$ with $b$ a positive integer $\geq 2$. For each such $b$, $u_b(n)$ appears in a rather interesting polynomial identity. Considering two special cases of this polynomial identity we find a new binomial identity and another identity which has application to the Tarry-Escott problem in number theory.

The function $u_2(n)$ also gives rise to a set of functions on $[0, 1]$ which constitute a lacunary subsequence of the Walsh functions and which have some properties similar to the Rademacher functions.

**1. The functions $u_b(n)$.** Let $\overline{G}$ be a commutative ring with unity such that there exists a mapping $f$ of $J^+$, the collection of nonnegative integers, into a subset $G$ of $\overline{G}$ satisfying $f(a+b)=f(a)f(b)$. Then when $b\geq 2$ we find that

$$\prod_{n=1}^{k}(1-f(b^{n-1}))^{b-1}=\sum_{n=1}^{b^k}u_b(n)f(n-1), \tag{1}$$

where $u_b(n)$ is $\pm 1$ times a product of binomial coefficients. In order to give an explicit expression for $u_b(n)$ we define two other functions of $n$.

$b_j(n)$ is the coefficient of $b^j$ in the expansion of $n$ to the base $b$,

$$v_b(n)=\sum_{j=0}^{\infty}b_j(n). \tag{2}$$

Using (2) we have

$$u_b(n)=(-1)^{v_b(n-1)}\prod_{j=0}^{\infty}\binom{b-1}{b_j(n-1)}. \tag{3}$$

Although $u_b(n)$ appears in (1) only for $1 \leq n \leq b^k$, the relation (3) defines $u_b(n)$ for all $n \geq 1$. Taking $1 \leq n \leq b^k$, $m \geq 0$ we find that $v_b(n-1+mb^k) = v_b(n-1)+v_b(m)$ and

$$\prod_{j=0}^{\infty}\binom{b-1}{b_j(n-1+mb^k)} = \prod_{j=0}^{\infty}\binom{b-1}{b_j(n-1)}\prod_{j=0}^{\infty}\binom{b-1}{b_j(m)}.$$

Hence from (3) we conclude that

$$u_b(n+mb^k) = u_b(n)u_b(m+1) \text{ for } 1 \leq n \leq b^k, \qquad m \geq 0. \tag{4}$$

Formula (4) enables us to calculate $u_b(n)$ inductively from $u_b(n)$ for $1 \leq n \leq b$ and these are readily determined from (3). We discuss the calculation of the $u_b(n)$ again in Section 3.

Taking $b=2$ we find $u_2(n) = (-1)^{v_2(n-1)}$ and therefore $u_2(n)$ is 1 or $-1$ according to whether the digit 1 appears an even or an odd number of times in the dyadic expansion of $n-1$. Writing $+$ for 1 and $-$ for $-1$ the sequence of $u_2(n)$ starts as follows:

(5) $+--+-++--++-+--+-++-+--++--+-++-$.

Note that the first $2^k$ digits are the negatives of the next following $2^k$ digits for all $k \geq 0$.

**2. Translation operators and a polynomial identity.** Let $D(a_1, \cdots, a_k; b_1, \cdots, b_k)$, $a_i$ and $b_i$ integers, be an operator which maps the polynomial $P(x)$ into the polynomial $a_1P(x+b_1)+ \cdots +a_kP(x+b_k)$. The totality of such operators is a commutative ring with unity. Letting this ring be $\overline{G}$ and taking $G$ to be the subset of "translation operators" $E(b) = D(1; b)$, $b$ a nonnegative integer, we find that the mapping $f$: $b \leftrightarrow E(b)$ from $J^+$ to $G$ has the property $f(a+b) = f(a)f(b)$. Hence (1) of Section 1 becomes

$$\prod_{n=1}^{k}(1 - E(b^{n-1}))^{b-1} = \sum_{n=1}^{b^k} u_b(n)E(n-1). \tag{6}$$

Noting that $1-E(a)$ maps constants onto zero and reduces the degree of all other polynomials by one we see that the operator in (6) will map all polynomials of degree less than $k(b-1)$ onto zero; *i.e.* $\sum_{n=1}^{b^k} u_b(n)E(n-1)P(x) = 0$ for $P(x)$ a polynomial of degree less than $k(b-1)$. This then yields the identity

$$\sum_{n=1}^{b^k} u_b(n)P(x+n-1) = 0 \text{ for } P(x) \text{ of degree less than } k(b-1). \tag{7}$$

Formula (7), with $x = q/m$, yields the binomial identity

$$\sum_{n=1}^{b^k} u_b(n)\binom{j+q+m(n-1)}{q} = 0, \qquad 0 \leq q < k(b-1), \qquad j \geq 0, \tag{8}$$

when $P(x)$ is taken to be the polynomial $(j+mx)\cdots(j+mx-q+1)/q!$ of degree $q$. This identity (8) is especially nice when $b=2$.

In terms of the Pascal triangle, identity (8), with $b=2$, says that if we start choosing numbers spaced $m$ rows apart ($m\geqq 0$) from the $q+1$st column, beginning anywhere ($j\geqq 0$), and take $2^k$ of them, where $k>q$, then if we append the first $2^k$ signs of (5) to the numbers obtained the resulting numbers sum to zero. For example, if $j=q=2$, $k=3$, $m=1$, we have $3-6-10+15-21+28+36-45=0$.

Taking $P(x)=(x+1)^m$, $0\leq m<k(b-1)$, in (7) gives

$$\sum_{n=1}^{b^k} u_b(n)(x+n)^m = 0. \tag{9}$$

From (9) we obtain the following two identities by letting $x$ be 0 and $q/p$, respectively.

$$\sum_{n=1}^{b^k} u_b(n)n^m = 0 \text{ for } 0\leq m<k(b-1). \tag{10}$$

$$\sum_{n=1}^{b^k} u_b(n)(q+pn)^m = 0 \text{ for } 0\leq m<k(b-1),\ p \text{ and } q \text{ arbitrary real numbers.} \tag{11}$$

**3. A modified Pascal triangle.** In this section we give a method for the rapid computation of the $u_b(n)$ and at the same time show in a more striking way the connection between the $u_b(n)$ and the binomial coefficients.

When $1\leq n\leq b$ equation (3) becomes

$$u_b(n) = (-1)^{n-1}\binom{b-1}{n-1}.$$

Therefore we can obtain the values of $u_b(n)$, $1\leq n\leq b$, by reading from left to right in the $b$th row of the following modified Pascal triangle.

$$\begin{array}{rrrrrrr}
1 \\
1 & -1 \\
1 & -2 & 1 \\
1 & -3 & 3 & -1 \\
1 & -4 & 6 & -4 & 1 \\
1 & -5 & 10 & -10 & 5 & -1 \\
1 & -6 & 15 & -20 & 15 & -6 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Now if we take $k=1$ in (4) we obtain $u_b(n+mb)=u_b(n)u_b(m+1)$ for $1\leq n\leq b$, $m\geq 0$. Successive applications of this formula enable us to extend to the right the rows of the above modified Pascal triangle thereby obtaining the $u_b(n)$ for $n>b$. Thus at the first step we multiply each of the first $b$ elements of the $b$th row ($b\geq 2$) by the 2nd element in that row and put these new numbers at the end of the row one after the other. The completed first step yields

$$
\begin{array}{rrrrrrrrrrrr}
1 \\
1 & -1 & -1 & 1 \\
1 & -2 & 1 & -2 & 4 & -2 \\
1 & -3 & 3 & -1 & -3 & 9 & -9 & 3 \\
1 & -4 & 6 & -4 & 1 & -4 & 16 & -24 & 16 & -4 \\
1 & -5 & 10 & -10 & 5 & -1 & -5 & 25 & -50 & 50 & -25 & 5 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

At the second step each of the first $b$ elements of the $b$th row is multiplied by the 3rd element in that row and the resulting numbers placed one by one at the end of the row. At the next step we multiply by the 4th element in each row, *etc.* The final resulting array is as follows.

$$
\begin{array}{rrrrrrrrrrrrrrrrl}
1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \cdots \\
1 & -2 & 1 & -2 & 4 & -2 & 1 & -2 & 1 & -2 & 4 & -2 & 4 & -8 & 4 & -2 & 4 \cdots \\
1 & -3 & 3 & -1 & -3 & 9 & -9 & 3 & 3 & -9 & 9 & -3 & -1 & 3 & -3 & 1 & -3 \cdots \\
1 & -4 & 6 & -4 & 1 & -4 & 16 & -24 & 16 & -4 & 6 & -24 & 36 & -24 & 6 & -4 & 16 \cdots \\
1 & -5 & 10 & -10 & 5 & -1 & -5 & 25 & -50 & 50 & -25 & 5 & 10 & -50 & 100 & -100 & 50 \cdots \\
1 & -6 & 15 & -20 & 15 & -6 & 1 & -6 & 36 & -90 & 120 & -90 & 36 & -6 & 15 & -90 & 225 \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots\cdot
\end{array}
$$

Thus this array of numbers contains all $u_b(n)$ for $b \geq 2$, $n \geq 1$.

**4. Sums of powers of integers.** The extension of the modified Pascal triangle discussed in the preceding section can be used to illuminate some of the results in Section 2. We consider here formula (10).

Since the $u_b(n)$ are just the numbers in the $b$th row of the final array of numbers given in Section 3 we have the following. Write below the numbers in the $b$th row of this array the successive $m$th powers of the integers. In the case $b=4$ we have

$$
\begin{array}{rrrrrrrrrrrrrrrrl}
1 & -3 & 3 & -1 & -3 & 9 & -9 & 3 & 3 & -9 & 9 & -3 & -1 & 3 & -3 & 1 & -3 \cdots \\
1^m & 2^m & 3^m & 4^m & 5^m & 6^m & 7^m & 8^m & 9^m & 10^m & 11^m & 12^m & 13^m & 14^m & 15^m & 16^m & 17^m \cdots.
\end{array}
$$

If we now multiply each number in the first row by the power directly below it in the second row and regard these products as summands then the sum of the first $b$ products is 0 when $0 \leq m < b-1$, the sum of the first $b^2$ products is 0 when $0 \leq m < 2(b-1)$, $\cdots$, the sum of the first $b^k$ products is 0 when $0 \leq m < k(b-1)$. In the example above with $b=4$ we have

$$1^m - 3 \cdot 2^m + 3 \cdot 3^m - 4^m = 0, \qquad 0 \leq m < 3,$$

$$1^m - 3 \cdot 2^m + 3 \cdot 3^m - 4^m - 3 \cdot 5^m + 9 \cdot 6^m - 9 \cdot 7^m + 3 \cdot 8^m + 3 \cdot 9^m - 9 \cdot 10^m$$
$$+ 9 \cdot 11^m - 3 \cdot 12^m - 13^m + 3 \cdot 14^m - 3 \cdot 15^m + 16^m = 0, \quad 0 \leq m < 6,$$
$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

In the next section we consider sums of powers of the above kind which arise from (10) in the case $b=2$.

**5. Application to the Tarry-Escott problem.** The Tarry-Escott problem is concerned with finding sets of integers $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ such that

$$\sum_{i=1}^{n} a_i^t = \sum_{i=1}^{n} b_i^t, \qquad 0 \le t \le k. \tag{12}$$

If (12) holds we shall write $a_1, \cdots, a_n \overset{k}{=} b_1, \cdots, b_n$.

From (10) in Section 2 we can obtain in the case $b=2$ the result

$$\sum_{n=1}^{2^{k+1}} u_2(n)n^m = 0 \text{ for } 0 \le m \le k. \tag{13}$$

Similarly from (11) we obtain

$$\sum_{n=1}^{2^{k+1}} u_2(n)(q + pn)^m = 0 \text{ for } 0 \le m \le k. \tag{14}$$

Using (13) we can give a very simple proof of the known proposition:

*For all $k \ge 0$ there exist sets of integers $a_i$ and $b_i$ such that $a_1, \cdots, a_n \overset{k}{=} b_1, \cdots, b_n$.*

We need only take $a_1, \cdots, a_n$ to be those integers between 1 and $2^{k+1}$ inclusive with $u_2(a_i)=1$ and $b_1, \cdots, b_n$ to be those with $u_2(b_i)=-1$. Then $a_1, \cdots, a_n \overset{k}{=} b_1, \cdots, b_n$ by (13).

The usual way of proving this result is to use the following proposition due to Tarry.

*If $a_1, \cdots, a_n \overset{k}{=} b_1, \cdots, b_n$ then, for all $h$,*

$$a_1, \cdots, a_n, b_1 + h, \cdots, b_n + h \overset{k+1}{=} b_1, \cdots, b_n, a_1 + h, \cdots, a_n + h.$$

The advantage of our proof is that it manages to get at the $a_i$ and $b_i$ directly without having to use a stepwise procedure.

Our solution for $k$ is, however, one possible result obtained by applying Tarry's theorem repeatedly and starting with $1 \overset{0}{=} 2$. Using Tarry's theorem with $h=2^k$ we deduce from the equation $\sum_{n=1}^{2^k} u_2(n)n^m=0$, valid for $0 \le m < k$, the equation

$$\sum_{n=1}^{2^k} \left\{ u_2(n)n^m + u_2(n + 2^k)(n + 2^k)^m \right\} = 0,$$

valid for $0 \le m \le k$. But the left side of this last equation is

$$\sum_{n=1}^{2^k} u_2(n)n^m + \sum_{n=2^k+1}^{2^{k+1}} u_2(n)n^m = \sum_{n=1}^{2^{k+1}} u_2(n)n^m,$$

and this is just the left side of our solution for $k$. Hence, starting with our solution for $k=0$ and applying Tarry's theorem $k$ times with $h=2, 2^2, 2^3, \cdots$, we arrive at $\sum_{n=1}^{2^{k+1}} u_2(n)n^m=0$.

Another known theorem is the following:

*Any set of $2^{k+1}$ integers in arithmetic progression can be split into equinumerous classes $a_1, \cdots, a_{2^k}$ and $b_1, \cdots, b_{2^k}$ such that $a_1, \cdots, a_{2^k} \stackrel{k}{=} b_1, \cdots, b_{2^k}$.*

We can generalize this theorem and at the same time make the splitting explicit by using (14). We get

*Any set of $2^{k+1}$ numbers in arithmetic progression can be split into equinumerous classes $a_1, \cdots, a_{2^k}$ and $b_1, \cdots, b_{2^k}$ such that $a_1, \cdots, a_{2^k} \stackrel{k}{=} b_1, \cdots, b_{2^k}$ and this splitting can be effected by* (14), *taking the $a_i$ to be the positive terms and the $b_i$ to be the negative terms.*

**6. An orthonormal set of functions.** Define the functions $u^{(n)}(x)$, $n \geq 0$, on $0 \leq x \leq 1$ by

$$u^{(n)}(x) = \begin{cases} 0 \text{ if } 2^{n+1}x \text{ is an integer,} \\ u_2(1 + [2^{n+1}x]) \text{ otherwise,} \end{cases} \tag{15}$$

where $[y]$ denotes the greatest integer $\leq y$. Defining the Rademacher functions $r_n(x)$, $n \geq 0$, on $0 \leq x \leq 1$ by

$$r_n(x) = \operatorname{sgn} \sin (2^{n+1}\pi x), \tag{16}$$

we can show that

$$u^{(n)}(x) = \prod_{i=0}^{n} r_i(x). \tag{17}$$

Noting that the Walsh functions $\psi_n(x)$, $n \geq 0$, on $0 \leq x \leq 1$ are defined by

$$\begin{aligned} &\psi_0(x) = 1, \\ &\psi_N(x) = \prod_{i=1}^{k} r_{n_i}(x), \qquad N = \sum_{i=1}^{k} 2^{n_i}, \end{aligned} \tag{18}$$

we see that, in virtue of (17), the functions $u^{(n)}(x)$ form a lacunary subsequence of the Walsh functions.

The theorem which states that $\sum_{i=1}^{\infty} a_i r_i(x)$ converges almost everywhere in $0 \leq x \leq 1$ when $\sum_{i=1}^{\infty} a_i^2 < \infty$ remains true when the $r_i(x)$ are replaced by $u^{(i)}(x)$ and with the same proof (6, pp. 126–7).

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# ON MORERA'S THEOREM

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The Theorem of Morera states that if $f$ is a single-valued, continuous function from a region $R$ of the $z$-plane to the complex numbers and if

$$\int_C f(z)dz = 0 \tag{1}$$

for every closed, rectifiable curve $C$ in $R$, then $f$ is holomorphic (that is, $f$ is analytic and regular) in $R$. It was early recognized that one need not assume (1) for *every* closed rectifiable curve in $C$ but only for some more restrictive class of curves. Osgood [**5**] gave a proof of Morera's theorem in which he assumed that (1) held on all "small" rectangles with horizontal and vertical edges lying within $R$. Rademacher [**6**] showed that the condition of continuity of $f$ in Osgood's theorem may be replaced by Lebesgue integrability over $R$ and linear integrability over horizontal and vertical segments lying in $R$. He then obtained that $f$ is almost everywhere equal to a holomorphic function in $R$.

Looman [**4, 9**] showed that, for continuous $f$, (1) may be replaced by the condition that at each point $z_0 \in R$,

$$\limsup_{m(Q)\to 0} \frac{1}{m(Q)} \left| \int_Q f(z)dz \right| = \sigma(z_0) < \infty, \tag{2}$$

and $\sigma(z_0) = 0$ for almost all $z_0 \in R$, where $Q$ represents a square with center $z_0$ and horizontal and vertical edges, and $m(Q)$, denotes the area enclosed by $Q$. Wolff [**10**] and Ridder [**8**] relaxed the hypotheses of Looman's theorem somewhat.

In all of these works, the squares or rectangles with horizontal and vertical edges played an essential role. It is well known that harmonic functions are characterized by the mean value property on small circles [**1**] and we may ask whether holomorphic functions are likewise characterized by the Morera theorem for small circles. The problem is different from that for rectangles since circles do not lend themselves to building up paths between two points. The method we use to prove the Morera theorem for "small" circles is that of smoothing operators or areal means [**2, 7**]. This application of the method affords an easy way to become acquainted with this important tool that is so often used now in existence proofs.

We first prove the type of theorem we seek under stronger hypotheses than necessary and later weaken these hypotheses.

THEOREM 1. *Let $f$ be a single-valued function from a region $R$ in the $z$-plane to the complex numbers and let $f \in C^1(R)$ (i.e., $f$ has continuous first partial derivatives, $\partial f/\partial x$ and $\partial f/\partial y$, in $R$). Assume that*

$$\lim_{r\to 0} \frac{1}{r}\int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta = \lim_{r\to 0}\frac{1}{r^2}\int_{C_r(z_0)} f(z)dz = 0 \tag{3}$$

*for each $z_0 \in R$, where $C_r(z_0)$ denotes the circle $|z-z_0| = r$. Then $f$ is holomorphic in $R$.*

We observe that if $f = u+iv$, $u$, $v$ real, then for any circle $C$ in $R$,

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (udy + vdx).$$

According to Green's theorem, we have

$$\int_C f(z)dz = -\iint_D \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)dxdy + i\iint_D\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy, \tag{4}$$

where $D$ represents the region enclosed by $C$. If $F$ is any continuous function in a neighborhood of $z_0$, then

$$\lim_{\rho\to 0}\frac{1}{\rho^2}\int_0^{\rho}\int_0^{2\pi} F(z_0 + re^{i\theta})rd\theta dr = \pi F(z_0).$$

Treating $\partial u/\partial y + \partial v/\partial x$ or $\partial u/\partial x - \partial v/\partial y$ as $F$, the relations (3) and (4) tell us that the Cauchy-Riemann equations hold for each $z_0 \in R$, so that $f$ is holomorphic in $R$.

COROLLARY 1.1. *If $f$ is a single-valued function in $R$ with $f \in C^1(R)$, and to each $z_0 \in R$ corresponds an $r_0$ such that*

$$\int_{C_r(z_0)} f(z)dz = 0 \tag{5}$$

*for $r < r_0$, then $f$ is holomorphic in $R$.*

Indeed, it is clear that (5) implies (3).

COROLLARY 1.2. *If the single-valued function $f \in C^1(R)$ satisfies at each point $z_0 \in R$,*

$$\lim_{\rho\to 0}\frac{1}{\rho^2}\int_0^{\rho}\int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta dr = 0, \tag{6}$$

*then $f$ is holomorphic in $R$.*

To prove Corollary 1.2, we set $G(r) = \int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta$ and proceed to show that

$$\lim_{r\to 0} \frac{1}{r} G(r) = 0, \tag{7}$$

when

$$\lim_{\rho\to 0} \frac{1}{\rho^2} \int_0^\rho G(r)dr = 0. \tag{8}$$

We observe first that $G(0)=0$ and that since $f\in C^1(R)$, $G(r)$ has a continuous derivative $G'$ for $0\leq r\leq r_0$ for some $r_0>0$. In fact, we have

$$G'(r) = \int_0^{2\pi} \frac{\partial f(z_0 + re^{i\theta})}{\partial r} e^{i\theta}d\theta.$$

Then $\lim_{r\to 0} G(r)/r = G'(0)$ while

$$\frac{1}{\rho^2}\int_0^\rho G(r)dr - \frac{1}{2}G'(0) = \frac{1}{\rho^2}\int_0^\rho \int_0^r [G'(s) - G'(0)]dsdr.$$

Since $G'$ is continuous at $r=0$, for a given $\epsilon>0$, there exists an $r_0>0$ such that $|G'(s)-G'(0)|<\epsilon$ when $0\leq s\leq r_0$. Thus if $\rho<r_0$, we have

$$\left|\frac{1}{\rho^2}\int_0^\rho G(r)dr - \frac{1}{2}G'(0)\right| < \frac{1}{2}\epsilon,$$

proving that

$$\lim_{\rho\to 0}\frac{1}{\rho^2}\int_0^\rho G(r)dr = \frac{1}{2}G'(0),$$

and (7) follows from (8). This implies that Corollary 1.2 follows from Theorem 1.

We now remove the differentiability hypothesis of Theorem 1 by introducing smoothing operators or areal means of a function. Let us first define

$$s_\rho(x+iy) = \begin{cases} k(\rho^2 - x^2 - y^2)^2 & \text{for } x^2+y^2<\rho^2, \\ 0 & \text{for } x^2+y^2\geq\rho^2, \end{cases} \tag{9}$$

where $k$ is chosen so that

$$\int_0^{2\pi}\int_0^\rho s_\rho(re^{i\theta})rdrd\theta = 2\pi k\int_0^\rho(\rho^2-r^2)^2rdr = 1.$$

We say that a function $f$ is Lebesgue integrable *in* a region $R$ if it is Lebesgue integrable over every compact subset of $R$. Let $R_\rho$ be the subset of $R$ consisting of those $z_0$ such that the entire disk $|z-z_0|\leq\rho$ lies in $R$. Then if $f$ is Lebesgue integrable in $R$, we define the areal mean of $f$ to be

$$(10)\qquad M(f, \rho; z_0) = \int_0^{2\pi} \int_0^{\rho} f(z_0 + re^{i\theta}) s_\rho(re^{i\theta}) r dr d\theta$$

for $z_0 \in R_\rho$. Since $s_\rho(z) = 0$ for $|z| \geq \rho$, we may define $f(z) = 0$ for $z$ in the complement of $R$ and note that (10) may also be written as

$$(11)\qquad M(f, \rho; z_0) = \iint_{|\zeta|<\infty} f(\zeta) s_\rho(\zeta - z_0) d\xi d\eta, \qquad \zeta = \xi + i\eta.$$

We next state and prove several properties of the operator $M$ defined in (10).

PROPERTY 1. *If $f$ is Lebesgue integrable in $R$, then $M(f, \rho; z)$ is in class $C^1$ in $R_\rho$ (i.e., $M(f, \rho; z)$ has continuous first partial derivatives in $R_\rho$).*

If we let $z = x + iy$, we may compute the difference quotient of $M(f, \rho; z)$ at the points $z$ and $z + \Delta x$. This is

$$\frac{M(f, \rho; z + \Delta x) - M(f, \rho; z)}{\Delta x} = \iint_{|\zeta|<\infty} f(\zeta) \left[ \frac{s_\rho(\zeta - z - \Delta x) - s_\rho(\zeta - z)}{\Delta x} \right] d\xi d\eta.$$

According to the mean value theorem applied to the function $s_\rho(\zeta - z)$, which has continuous partial derivatives in the whole plane, this may be rewritten as

$$\frac{M(f, \rho; z + \Delta x) - M(f, \rho; z)}{\Delta x} = \iint_{|\zeta|<\infty} f(\zeta) \frac{\partial}{\partial x} \left\{ s_\rho(\zeta - z - \theta \Delta x) \right\} d\xi d\eta,$$

where $0 < \theta < 1$. The function $\partial s_\rho(z)/\partial x$ vanishes for $|z| > \rho$, and being continuous, it is uniformly bounded in the whole plane. Thus, we may apply the Lebesgue convergence theorem as $\Delta x \to 0$ and obtain

$$\frac{\partial M(f, \rho, z)}{\partial x} = \iint_{|\zeta|<\infty} f(\zeta) \frac{\partial s_\rho(\zeta - z)}{\partial x} d\xi d\eta.$$

A similar expression holds for $\partial M(f, \rho, z)/\partial y$, which proves Property 1.

PROPERTY 2. *If $f$ is holomorphic in $R$ and $z_0 \in R_\rho$, then,*

$$(12)\qquad M(f, \rho; z_0) = f(z_0).$$

For any holomorphic function $f$ and any $z$ in $|z - z_0| \leq \rho$, we have $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and

$$\int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta = \sum_{n=0}^{\infty} a_n r^n \int_0^{2\pi} e^{in\theta} d\theta = 2\pi f(z_0),$$

for all $r \leq \rho$. From this we conclude that

$$M(f, \rho; z_0) = \int_0^{\rho} \int_0^{2\pi} f(z_0 + re^{i\theta}) s_\rho(re^{i\theta}) r d\theta dr$$

$$= k \int_0^\rho r(\rho^2 - r^2)^2 dr \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

$$= 2\pi f(z_0) k \int_0^\rho r(\rho^2 - r^2)^2 dr = f(z_0).$$

PROPERTY 3. *If $f$ is continuous in $R$, then* $\lim_{\rho\to 0} M(f, \rho; z) = f(z)$ *uniformly on any compact subset of $R$.*

For $n = 1, 2, \cdots$, let $\tilde{R}_n$ denote those points of $|z| \leq n$ which are interior to $R$ and at a distance greater than or equal to $1/n$ from the boundary of $R$. Then $\tilde{R}_n$ is compact, $\tilde{R}_n \subset \tilde{R}_{n+1}$, and $R = \bigcup_{n=1}^{\infty} \tilde{R}_n$. It suffices to prove Property 3 for the sets $\tilde{R}_n$. For any $z_0 \in \tilde{R}_n$, the disk $|z - z_0| \leq \rho$ lies entirely within $\tilde{R}_{2n}$ whenever $\rho < 1/(2n)$. Then

$$f(z_0) - M(f, \rho; z_0) = \int_0^{2\pi} \int_0^\rho [f(z_0) - f(z_0 + re^{i\theta})] s_\rho(re^{i\theta}) r dr d\theta.$$

Since $\tilde{R}_{2n}$ is compact, $f$ is uniformly continuous on $\tilde{R}_{2n}$ and given $\epsilon > 0$, there exists a $\delta > 0$ such that when $|z_1 - z_2| < \delta$, $z_1, z_2 \in \tilde{R}_{2n}$, we have $|f(z_1) - f(z_2)| < \epsilon$. Thus when $\rho < \delta$,

$$| f(z_0) - M(f, \rho; z_0) | \leq \epsilon \int_0^{2\pi} \int_0^\rho s_\rho(re^{i\theta}) r dr d\theta = \epsilon,$$

which proves Property 3.

PROPERTY 4. *If $f$ is Lebesgue integrable in $R$, then*

$$\lim_{\rho \to 0} \iint_D | f(z) - M(f, \rho; z) | \, dxdy = 0$$

*for any compact subset $D$ of $R$.*

It suffices to prove this theorem for the $\tilde{R}_n$ defined in the proof of Property 3. The function $f$ may be approximated "in the mean" by continuous functions on $\tilde{R}_n$; *i.e.*, given $\epsilon > 0$, there is a continuous function $g$ on $\tilde{R}_n$ such that

$$\iint_{\tilde{R}_n} | f(z) - g(z) | \, dxdy < \epsilon.$$

Then we have for $\tilde{R}_n$

$$\iint_{\tilde{R}_n} | f(z) - M(f, \rho; z) | \, dxdy \leq \iint_{\tilde{R}_n} | f(z) - g(z) | \, dxdy$$

$$+ \iint_{\tilde{R}_n} | g(z) - M(g, \rho; z) | \, dxdy + \iint_{\tilde{R}_n} | M(g - f, \rho; z) | \, dxdy.$$

The first integral on the right is less than $\epsilon$ by the choice of $g$. The second integral is made less than $\epsilon$ by making $\rho$ small enough, say $\rho<\rho_0$ (Property 3). It remains to prove that the third integral is also small.

By Fubini's theorem, we may interchange the orders of integration over $\tilde{R}_n$ and over the disk of radius $\rho$. Thus

$$\begin{aligned}\iint_{\tilde{R}_n} | M(g-f), \rho; z) | \, dxdy & \\ &\leq \iint_{\tilde{R}_n}\left[\iint_{|\zeta|<\rho} | g(\zeta+z)-f(\zeta+z) | \, s_\rho(\zeta)d\xi d\eta\right]dxdy \\ &= \iint_{|\zeta|<\rho} s_\rho(\zeta)d\xi d\eta \iint_{\tilde{R}_n} | g(z+\zeta)-f(z+\zeta) | \, dxdy \\ &\leq \epsilon \iint_{|\zeta|<\rho} s_\rho(\zeta)d\xi d\eta = \epsilon.\end{aligned}$$

This proves Property 4.

THEOREM 2. *Let $f$ be a single-valued function in a region $R$ such that $f$ is Lebesgue integrable over any compact subset of $R$. Assume that for all $z\in R$,*

$$\lim_{\rho\to 0}\frac{1}{\rho^2}\int_0^\rho\int_0^{2\pi} f(z+re^{i\theta})e^{i\theta}d\theta dr = 0. \tag{13}$$

*Moreover, assume that for each compact subset $K\subset R$, there exist positive numbers $\rho_0$ and $M$ such that for any $z\in K$ and $\rho<\rho_0$,*

$$\frac{1}{\rho^2}\left|\int_0^\rho\int_0^{2\pi} f(z+re^{i\theta})e^{i\theta}d\theta dr\right| < M. \tag{14}$$

*Then $f$ is almost everywhere equal to a holomorphic function in $R$.*

Let us introduce the notation

$$A(f,\rho;z) = \int_0^\rho\int_0^{2\pi} e^{i\theta}f(z+re^{i\theta})d\theta dr. \tag{15}$$

We first show that the operators $A$ and $M$ commute; *i.e.*, given an integer $n$, then for $0<\sigma<1/(2n)$ and $0<\tau<1/(2n)$, we have

$$M(A(f,\sigma),\tau;z_0) = A(M(f,\tau),\sigma;z_0) \tag{16}$$

for all $z_0\in\tilde{R}_n$. (Here $A(f,\sigma)$ denotes the function whose value at $z$ is $A(f,\sigma;z)$ and similarly for $M(f,\tau)$). This is an immediate consequence of Fubini's theorem, for

$$M(A(f,\sigma),\tau;z_0) = \iint_{|\zeta|<\tau} s_\tau(\zeta)d\xi d\eta\int_0^\sigma\int_0^{2\pi} f(z_0+\zeta+re^{i\theta})e^{i\theta}d\theta dr$$

$$= \int_0^\sigma \int_0^{2\pi} e^{i\theta} d\theta dr \iint_{|\zeta|<\tau} f(z_0 + \zeta + re^{i\theta}) s_\tau(\zeta) d\xi d\eta = A(M(f, \tau), \sigma; z_0).$$

The function $\rho^{-2}A(f, \rho)$ satisfies (according to (13), (14)) $\lim_{\rho\to 0} \rho^{-2}A(f, \rho; z) = 0$ and corresponding to the compact set $\bar{R}_{2n}$, there exist an $M$ and $\rho_0 > 0$ such that $|\rho^{-2}A(f, \rho; z)| < M$ when $\rho < \rho_0$, $z \in \bar{R}_{2n}$. Since for $\tau < 1/(2n)$, $\rho^{-2} A(M(f, \tau), \rho; z_0) = M(\rho^{-2} A(f, \rho), \tau; z_0)$ for $z_0 \in \bar{R}_n$, we may apply the Lebesgue bounded convergence theorem to conclude that

$$\lim_{\rho\to 0} \frac{1}{\rho^2} (A(M(f, \tau), \rho; z_0) = 0$$

for all $z_0 \in \bar{R}_n$.

We know, however, that $M(f, \tau) \in C^1$ in $\bar{R}_n$ when $\tau < 1/(2n)$. According to Corollary 1.2, $M(f, \tau)$ is holomorphic in $\bar{R}_n$. This leaves us with the task of showing that $M(f, \tau)$ being holomorphic implies that $f$ is almost everywhere equal to a holomorphic function in $\bar{R}_n$.

We begin by showing that $M(f, \tau; z)$ is independent of $\tau$; *i.e.*, if $\tau, \sigma < 1/(2n)$, then

$$M(f, \tau; z) = M(f, \sigma; z) \tag{17}$$

for $z \in \bar{R}_{n/2}$. From Property 2 of $M$, we deduce that

$$M(M(f, \tau), \sigma; z) = M(f, \tau; z) \tag{18}$$

and

$$M(M(f, \sigma), \tau; z) = M(f, \sigma; z) \tag{19}$$

for $z \in \bar{R}_{n/2}$ and $\sigma, \tau < 1/(2n)$. But an application of Fubini's theorem shows that the left sides of (18) and (19) are equal; for, if $\lambda = \mu + i\nu$,

$$M(M(f, \tau), \sigma; z) = \iint_{|\zeta|<\sigma} s_\sigma(\zeta) d\xi d\xi \iint_{|\lambda|<\tau} s_\tau(\lambda) f(z + \zeta + \lambda) d\mu d\nu$$

$$= \iint_{|\lambda|<\tau} s_\tau(\lambda) d\mu d\nu \iint_{|\zeta|<\sigma} s_\sigma(\zeta) f(z + \zeta + \lambda) d\xi d\eta = M(M(f, \sigma), \tau; z).$$

Thus (17) is established.

We observe next that according to Property 4 of $M$,

$$\lim_{\sigma\to 0} \iint_{R_{n/2}} | M(f, \sigma; z) - f(z) | dxdy = 0.$$

Since $M(f, \sigma; z) = M(f, \tau; z)$ for fixed $\tau$ as $\sigma \to 0$, we see that

$$\iint_{R_{n/2}} | M(f, \tau; z) - f(z) | dxdy = 0$$

and finally $f(z) = M(f, \tau; z)$ almost everywhere in $\bar{R}_{n/2}$. This tells us that $f$ is almost everywhere equal to a holomorphic function in $\bar{R}_{n/2}$, and since $n$ is arbitrary, the result holds for $R$.

When $f$ is Lebesgue integrable in $R$, then for each $z_0 \in R$ and any disk $|z-z_0| \leq \rho$ contained in $R$, we know from Fubini's theorem [3] that the integral

$$\int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta \tag{20}$$

exists for almost all $r$ in the interval $0 \leq r \leq \rho$. Let $E_\rho(z_0)$ denote the subset of $0 \leq r \leq \rho$ for which (20) exists. Then $E_\rho(z_0)$ has linear Lebesgue measure equal to $\rho$. We may now state the following corollary to Theorem 2.

Corollary 2.1. *Let $f$ be a single-valued function in a region $R$ such that $f$ is Lebesgue integrable over any compact subset of $R$. Assume that for $r \in E_\rho(z_0)$*

$$\lim_{r\to 0} \frac{1}{r} \int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta = 0 \tag{21}$$

*for all $z_0 \in R$. Furthermore, assume that to each compact set $K \subset R$, there correspond $M$ and $r_0$ such that*

$$\frac{1}{r}\left| \int_0^{2\pi} f(z + re^{i\theta})e^{i\theta}d\theta \right| < M \tag{22}$$

*for all $z \in K$ and almost all $r$ in the interval $0 \leq r \leq r_0$. Then $f$ is almost everywhere equal to a holomorphic function in $R$.*

It is clear that (21) and (22) imply (13) and (14) respectively. For example, if (21) holds, given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta \right| < \epsilon r$$

for almost all $r < \delta$. Then for $\rho < \delta$,

$$\left| \int_0^{\rho} \int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta dr \right| \leq \left( \int_0^{\rho} \epsilon r dr \right) (\epsilon\rho^2/2),$$

so that

$$\left| \frac{1}{\rho^2} \int_0^{\rho} \int_0^{2\pi} f(z_0 + re^{i\theta})e^{i\theta}d\theta dr \right| < \epsilon/2,$$

which in turn implies (13).

A weaker but interesting consequence of Corollary 2.1 is the following.

COROLLARY 2.2. *Let $f$ be a continuous, single-valued function in $R$ and let $\{U_i\}$ be an arbitrary open covering of $R$. Assume that*

$$\int_C f(z)dz = 0 \tag{23}$$

*for every circle $C$ that lies entirely within at least one open set of the covering $\{U_i\}$. Then $f$ is holomorphic in $R$.*

To prove this, we observe that (23) implies both (21) and (22). Since $f$ is assumed to be continuous, the conclusion of Corollary 2.1 may be replaced by the stronger statement that $f$ is everywhere holomorphic in $R$.

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# ON FLETCHER'S PAPER "CAMPANOLOGICAL GROUPS"

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In the article *Campanological Groups* that appeared in this MONTHLY, T. J. Fletcher [**1**] considered, among other things, Thompson's [**3**] solution to the question of whether the full peal of Grandsire Triples could be generated by the use of plain and bob leads alone. Fletcher continues: "Thompson shows this to be impossible by proving that plaining or bobbing any $Q$-set always results in the loss or gain of an even number of round blocks. . . . The beauty of the proof is marred by the fact that the stage showing that the number of round blocks lost or gained is always even, is carried out by a long and tedious process of enumeration. But it is very difficult to see any means by which this could have been avoided. The enumeration of the cosets of a group of large order is inevitably tedious, and modern processes do not seem to offer any way of reducing

Thompson's labors to any marked extent."

It is the purpose of this note to supply that portion of the proof that Thompson and Fletcher were seeking. The notation used is that of Fletcher [1]. The proof is derived from a theorem of Rankin [2] who proved a more general result, of which this is a special case.

Suppose that we have a decomposition $R_1$ of the 360 leads into round blocks and a $Q$-set $\{x(PB^{-1})^i\}$, $i=1, 2, 3, 4, 5$, each of whose members are bobbed. Let $S_1$ be the substitution on the numbers $i$ such that $j$ is substituted for $i$ if $x(PB^{-1})^j$ is the first member of the $Q$-set that occurs after $x(PB^{-1})^i$ in that round block of $R_1$ in which they occur. The number of cycles in the substitution $S_1$ is obviously the number of round blocks of $R_1$ that contain elements of the $Q$-set under consideration.

In $R_1$, the element that follows $x(PB^{-1})^i$ is $x(PB^{-1})^iB$. Now if $x(PB^{-1})^i$ is plained instead of bobbed, the element following becomes $x(PB^{-1})^iP$ $=x(PB^{-1})^{i+1}(PB^{-1})^{-1}P=x(PB^{-1})^{i+1}(BP^{-1})P=x(PB^{-1})^{i+1}B$. Hence we may form from $R_1$ a new decomposition $R_2$ by replacing the succession of $x(PB^{-1})^i$ by $x(PB^{-1})^iB$ by the succession of $x(PB^{-1})^i$ by $x(PB^{-1})^iP=x(PB^{-1})^{i+1}B$ and by letting all the other successions remain fixed. With respect to this new decomposition $R_2$ which was formed from $R_1$ by replacing a bobbed $Q$-set by a plained $Q$-set, and with respect to this $Q$-set, let $S_2$ be defined as $S_1$ was with respect to $R_1$.

Now it remains to show that the number of cycles of $S_1$ differs, if at all, from the number of cycles of $S_2$ by an even number. It is apparent that $S_2$ is the cyclical permutation (12345) followed by $S_1$. If we write the cyclical permutation as a product of transpositions, we have $(12)(13)(14)(15)S_1=S_2$. But multiplication of cycles by the transposition $(pq)$ increases the number of cycles by one if $p$ and $q$ are in the same cycle and diminishes the number of cycles by one if $p$ and $q$ are in different cycles. Since we have four transpositions, the number of round blocks with the chosen $Q$-set bobbed hence differs by an even number from the number of round blocks that have the $Q$-set plained.

The smallest round block is that formed by a succession of three bobbed leads. We have shown therefore that the largest round block formed by bobbing and plaining alone cannot exceed 357 leads or 4998 changes. That a touch of this length is actually attainable was shown in 1751 by John Holt [4].

I should like to remark that the literature dealing with change ringing is quite large and that some of the modern publications can be found by consulting the weekly journal of the change ringers [5].

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# CURVES WITH A KIND OF CONSTANT WIDTH

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Consider a simple, closed, plane, convex curve $C$ with the point $A$ in its interior. If $m$ is any line through $A$, there are two lines, $t$ and $\bar{t}$, which are supporting lines of $C$, at $R$ and $\bar{R}$, and which are perpendicular to $m$ at points $Q$ and $\bar{Q}$ respectively (Fig. 1). As $m$ turns about $A$, the point $Q$ traces a closed curve $C'$ (also traced by $\bar{Q}$), which is the pedal curve to $C$ with respect to the pedal point $A$.* The curve $C'$ is sometimes denoted by $P(C)$, and the curve $C$, which is the negative pedal curve to $C'$, is also denoted by $P^{-1}(C)$. The width of the curve $C$, in either direction along $m$, is defined to be the distance between the

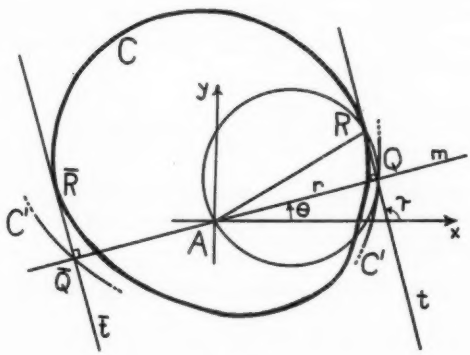


FIG. 1

parallel lines $t$ and $\bar{t}$, and is therefore the length of the chord from $Q$ to $\bar{Q}$ in $C'$. In particular, if $C$ is of constant width then all chords of $C'$ through $A$ have the same length, that is, $A$ is an *equichordal* point of $C'$.

The previous relationships suggest the method, first given by G. Tiercy,† of determining a constant width curve $C$ as the negative pedal of a curve $C'$ which has the pedal point as an equichordal point. If the (polar) equation of $C'$ is $r=f(\theta)$, where $A$ is the origin, then a supporting line to $C$ has the normal form equation,

$$x \cos \theta + y \sin \theta = f(\theta). \tag{1}$$

When $C$ is differentiable, it is the envelope of the family of lines given by (1). Regarding $\theta$ as a parameter, partial differentiation of (1) yields

* The notion of a pedal curve was introduced by C. Maclaurin (1698–1746).

† G. Tiercy, Sur les spheriformes, Tôhoku Math. J., vol. 19, 1921.

$$-x \sin \theta + y \cos \theta = f'(\theta). \tag{2}$$

From (1) and (2), the parametric equations for $C$ are,

$$\begin{aligned} x &= f(\theta) \cos \theta - f'(\theta) \sin \theta, \\ y &= f(\theta) \sin \theta + f'(\theta) \cos \theta. \end{aligned} \tag{3}$$

The condition that $C'$ have the origin as an equichordal point is simply that $f(\theta)+f(\theta+\pi)$ be constant for all $\theta$. It is apparent from the mechanical consideration of a stick rotating and sliding about a fixed point that there are an unlimited number of such curves. They can be constructed in a great variety of ways, even with the extra condition of convexity, for if $r=f(\theta)$ and $r=g(\theta)$ are equichordal at $A$ and are convex, then so is the curve $r=\alpha f(\theta)+\beta g(\theta)$, where $\alpha$ and $\beta$ are positive constants. The two properties are also preserved by a rotation at $A$. Starting with one such curve, $r=f(\theta)$, one can therefore rotate it and then form a positive, linear combination of the two curves. This process can be carried to the limit by integration. That is, if $h(\gamma)$ is any nonnegative function, not identically zero, then the function

$$p(\theta) = \int_0^{2\pi} f(\theta + \gamma) h(\gamma) d\gamma,$$

is convex and is equichordal at $A$. In an analytic form, a simple class of curves which are equichordal at the origin can be expressed by a general, odd term, Fourier series, that is, by

$$r = f(\theta) = a + a_1 \cos \theta + a_2 \cos 3\theta + \cdots + a_n \cos (2n - 1)\theta + \cdots, \tag{4}$$

where $a>0$. In particular, if (4) is specialized to the limaçon, $r=a+a_1 \cos \theta$, $a_1<a$, then (3) becomes $x=a_1+a \cos \theta$, $y=a \sin \theta$, which is the circle, $(x-a_1)^2+y^2=a^2$. This is a special case of the fact that, for a finite $n$ in (4), the curves (3) are rational and algebraic.

In order for the curve $C$ to be of constant width, it must of course be convex, and a sufficient condition for this is that it have nonnegative curvature at all its points. With the tangent to $C$ expressed by (1), it is a standard formula that the radius of curvature of $C$ is given by $f(\theta)+f''(\theta)$. Thus Tiercy imposes the extra condition on $C'$ that $r+r''\geqq 0$, and this, with the equichordal property, suffices to make $C$ a constant width curve.

The case, which Tiercy did not take up, that in which $r+r''$ changes sign, is also interesting. Assume that $C'$ is star shaped and equichordal with respect to an interior origin. If the tangent line to $C=P^{-1}(C')$ has $\tau$ as an angle of inclination, then, as in Figure 1, $\tau=\theta\pm 90°$, and $d\tau/d\theta=1$. Hence as $\theta$ increases monotonically in a circuit of $C'$, the tangent line to $C$ turns in a counterclockwise sense and there are exactly two parallel tangents to $C$ in any given direction. As long as $r+r''\geqq 0$, the point $R$ on $C$, corresponding to $Q$ on $C'$, traverses a convex arc. When, however, $r+r''$ changes to negative values, there is a cusp point on $C$, the radius vector to $R$ on $C$ reverses its sense of rotation, and $Q$ traverses a concave arc. With the next change of curvature there is another cusp, the rotation of the radius vector to $R$ again changes, and the curve $C$ crosses itself. Since $C$ is a closed curve, there will be an even number of these cusps. The sketch in Figure 2 illustrates such a curve $C$, where $C'$ has the form $r=3+2\cos^3\theta$. The parametric equations for $C$, given by (3), are

$$x = -4\cos^4\theta + 6\cos^2\theta + 3\cos\theta,$$
$$y = 3\sin\theta - 4\sin\theta\cos^3\theta, \tag{5}$$

and the two tangents in any given direction are six units apart. A closed curve having but one tangent in each direction has been called a curve of zero width. In a similar sense, the curve of Figure 2 has a kind of constant width.

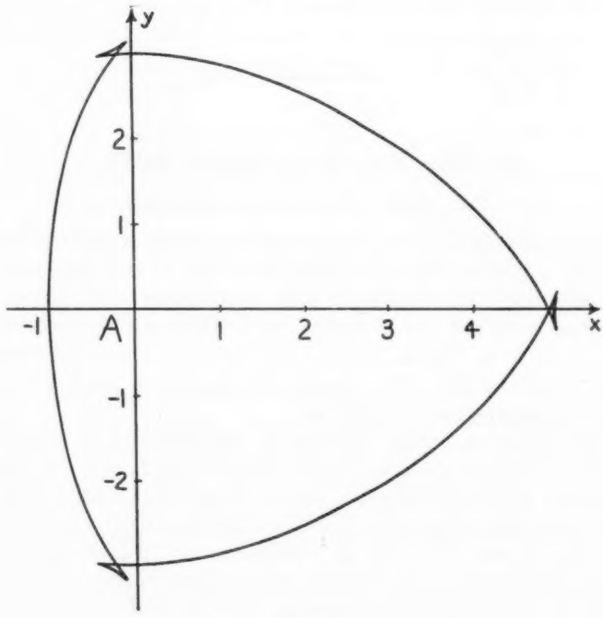


FIG. 2

Because it is the convexity or nonconvexity of $C$ which distinguishes the two types of constant width curves, it is natural to ask the following general question. For a given curve $C'$ what are the possible positions of a pedal point $A$ such that the negative pedal curve of $C'$ with respect to $A$ will be convex? It can be shown easily that $C$ will be convex if, and only if $C^*$ is convex, where $C^*$ is obtained from $C'$ by an inversion with respect to a circle centered at $A$.

In another place, Ernst Straus and the author have given general conditions for the inversive convexity of $C'$, and hence conditions also for the convexity of $C=P^{-1}(C')$.

For the special case considered here, that is when $C'$ has curvature and $A$ is chosen interior to $C'$, one can describe the convexity of $C$ in the following way. Let $K$ denote the circle of curvature of $C'$ at a point $Q$. As $Q$ moves about $C'$ in a circuit, so does $K$. So long as the origin, namely the interior pedal point, is within, or on $K$, then the point $R$ on $C$, corresponding to $Q$ on $C'$, traverses a convex arc, and when $A$ is outside $K$ then $R$ is on a concave arc of $C$. Thus, the cusps of Figure 2 correspond to $A$ passing in and out of the curvature circles of $C'$, and one can see from this that there will be an even number of cusps. A standard, pedal curve relationship is that the circle on $\overline{AR}$ as a diameter is always tangent to $C'$ at $Q$ (Figure 1). When $R$ is a cusp point of $C$, this circle is also a circle of curvature of $C'$.

# FILTERS AND EQUIVALENT NETS

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The purpose of this note is to introduce a natural equivalence relation between nets [**5**] in such a way as to bring filters [**3, 4**] and classes of equivalent nets into one-to-one correspondence. Our equivalence relation for nets stems from a theorem of Bruns and Schmidt [**2**, p. 184], while our correspondence rests on results of R. G. Bartle [**1**]. We feel that our discussion helps to clarify the concept of subnet [**5**]. The present self-contained version of our note is presented at the suggestion of a referee.

Let us first agree on certain matters of notation and of terminology. The notation $\alpha$: $S\rightarrow T$ denotes a map $\alpha$ of a set $S$ into a set $T$, $\alpha(s)$ denotes the image of $s\in S$ under $\alpha$, and $\alpha(S_1)$ denotes the set of all $\alpha(s_1)$ with $s_1\in S_1$. We write $\alpha\circ\beta$ for the composite of $\alpha$ and $\beta$: $T\rightarrow U$ so that $(\alpha\circ\beta)(s)=\alpha(\beta(s))$. A relation $\leqq$ *partially orders* a set $P$ in case $\leqq$ is reflexive and transitive. In a partially ordered set $P$, we put $p^+=[q\in P: q\geqq p]$ for each $p\in P$, and we call $P$ a *directed set* in case $p^+\cap q^+$ is non-empty for every $p, q\in P$. A family $\mathfrak{B}$ of non-empty subsets of a set $X$ which is directed by inverse set-inclusion is called a *filter-base* in $X$. The totality of filter-bases in $X$ will be denoted by $\mathfrak{G}(X)$. If $\mathfrak{B}$, $\mathfrak{D}\in\mathfrak{G}(X)$, then $\mathfrak{D}$ *refines* $\mathfrak{B}$ (which we write $\mathfrak{D}>\mathfrak{B}$) in case every set in $\mathfrak{B}$ contains some set in $\mathfrak{D}$. When $\mathfrak{D}>\mathfrak{B}$ and $\mathfrak{B}>\mathfrak{D}$, we call $\mathfrak{B}$ and $\mathfrak{D}$ *equivalent* and write $\mathfrak{B}\sim\mathfrak{D}$. We call $\mathfrak{B}$, $\mathfrak{D}\in\mathfrak{G}(X)$ *compositive* in case $\mathfrak{B}\vee\mathfrak{D}=[F\cap G: F\in\mathfrak{B}, G\in\mathfrak{D}]$ is a filter base in $X$. (Thus $\mathfrak{G}(X)$ is a partially ordered set which is not a directed set unless $X$ has only one element. We have borrowed the term *compositive* from the work of E. H. Moore [**7**] since we feel that this important relation between filter-bases deserves a name.) A *filter* in $X$ is a filter-base in $X$ which contains all supersets of each of its members. A *net* in $X$ is a map $\alpha: A \to X$, where $A$ is a directed set. A net $\beta: B \to X$ is a *subnet* of a net $\alpha$ in case $\beta = \alpha \circ \pi$, where $\pi: B \to A$ is *convergent* in the sense of E. H. Moore [**7**, p. 34], *i.e.*, there is a map $\rho: A \to B$ such that $\pi((\rho(a)^+) \subseteq a^+$ for every $a \in A$.

Each net $\alpha: A \to X$ gives rise to a filter-base $\mathbf{B}(\alpha) = [\alpha(a^+): a \in A]$ in $X$. We write $\mathbf{F}(\alpha)$ for the filter based on $\mathbf{B}(\alpha)$, *i.e.*, $\mathbf{F}(\alpha)$ is the family of all supersets of members of $\mathbf{B}(\alpha)$. If $\mathfrak{B} \in \mathfrak{G}(X)$, we may define $A(\mathfrak{B}) = [(x, F): x \in F \in \mathfrak{B}]$, $(x, F) \leqq (x_1, F_1)$ in case $F \subseteq F_1$, and $\beta((x, F)) = x$ to obtain a net $\beta = \mathbf{N}(\mathfrak{B})$ in $X$. (This is the substance of the footnote on p. 554 of [**1**] as well as of 2.$L(f)$ of [**6**].) It is easy to verify that $\mathbf{B}(\mathbf{N}(\mathfrak{B})) = \mathfrak{B}$ for every $\mathfrak{B} \in \mathfrak{G}(X)$.

A *topology* in a set $X$ is specified by a map $\tau: X \to \mathfrak{G}(X)$ such that $x \in U$ for every $U \in \tau(x)$. A point $x_0 \in X$ *$\tau$-adheres to a filter-base* $\mathfrak{B}$ in $X$ in case $\tau(x_0)$ and $\mathfrak{B}$ are compositive. A point $x_0 \in X$ *$\tau$-adheres to a net* $\alpha$ in $X$ in case $x_0$ $\tau$-adheres to $\mathbf{F}(\alpha)$, or, equivalently, to $\mathbf{B}(\alpha)$. We shall call a net $\alpha$ in $X$ *as fine as* a net $\beta$ in $X$ and write $\alpha \geqq \beta$ in case every point of $X$ which $\tau$-adheres to $\beta$ also $\tau$-adheres to $\alpha$ for every topology $\tau$ of $X$.

LEMMA. *For nets $\alpha$, $\beta$ in $X$, we have $\alpha \geqq \beta$ if and only if $\mathbf{F}(\alpha) \subseteq \mathbf{F}(\beta)$.*

*Proof.* Let us assume that $\alpha \geqq \beta$ and suppose that some $F \in \mathbf{F}(\alpha)$ is not in $\mathbf{F}(\beta)$. Then $F' \cap G \neq \phi$ for every $G \in \mathbf{F}(\beta)$, since $F' \cap G = \phi$ yields $G \subseteq F$, $F \in \mathbf{F}(\beta)$, a contradiction. We obtain a topology in $X$ by setting $\tau(x) = [[x]]$ for every $x \in F$ and $\tau(x) = [F' \cap G: G \in \mathbf{F}(\beta)]$ for $x \in F'$. But then every $x$ in $F'$ $\tau$-adheres to $\beta$, while no $x$ in $F'$ $\tau$-adheres to $\alpha$, contrary to our assumption, $\alpha \geqq \beta$. We have proved that $\alpha \geqq \beta$ implies $F(\alpha) \subseteq F(\beta)$. The converse is trivial. The proof of the lemma is complete.

It is now easy to see that $\alpha \geqq \beta$ if and only if $\mathbf{B}(\beta) \succ \mathbf{B}(\alpha)$. It is also clear that if $\alpha \geqq \beta$ and $\eta: X \to Y$ is a map of $X$ into $Y$, then $\eta \circ \alpha$ and $\eta \circ \beta$ are nets in $Y$ such that $\eta \circ \alpha \geqq \eta \circ \beta$. Thus the convergent maps of E. H. Moore are just those maps $\pi: B \to A$ for which $e_A \geqq \pi$, where $e_A$ is the identity map of $A$ onto $A$. The subnets $\beta = \alpha \circ \pi$ of $\alpha$ then satisfy $\alpha \geqq \beta$ but, even for finite $X$, it is possible that $\alpha \geqq \beta$ while $\beta$ is not a subnet of $\alpha$. (See, however, the remark contained in our final paragraph.)

It seems reasonable, now, to call two nets $\alpha$, $\beta$ in $X$ *equivalent* and to write $\alpha \sim \beta$ in case $\alpha \geqq \beta$ and $\beta \geqq \alpha$, *i.e.*, $\alpha \sim \beta$ if and only if $\mathbf{F}(\alpha) = \mathbf{F}(\beta)$, $\alpha \sim \beta$ if and only if $\mathbf{B}(\alpha) \sim \mathbf{B}(\beta)$, since when this is true there is no topology in $X$ which will distinguish between them. Using this notation, we have $\mathbf{N}(\mathbf{B}(\alpha)) \sim \alpha$ for every net $\alpha$ in $X$ because $\mathbf{B}(\mathbf{N}(\mathbf{B}(\alpha))) = \mathbf{B}(\alpha)$. The maps $\mathbf{B}$ and $\mathbf{N}$ establish a one-to-one correspondence between the classes of equivalent nets in $X$ and the classes of equivalent filter-bases in $X$; while the maps $\mathbf{F}$ and $\mathbf{B}$ serve the same purpose for filters in $X$ and classes of equivalent nets in $X$.

Let us add one further observation concerning subnets. In the present notation, Kelley's fundamental lemma on subnets [**5**, p. 278] can be given a slightly more precise form. Let $\alpha$ be a net in $X$, $\mathfrak{B}$ a filter-base in $X$ compositive with $\mathbf{B}(\alpha)$. Let $\gamma((x, a, B))=x$ for $a\in A$, $B\in\mathfrak{B}$, and $x\in\alpha(a^+)\cap B$. Define $(x, a, B) \leqq (y, a_1, B_1)$ in case $a\leqq a_1$ and $B_1\subseteq B$. Then it is easy to see that $\gamma$ is a subnet of $\alpha$ such that $\mathbf{B}(\gamma)=\mathfrak{B}\vee\mathbf{B}(\alpha)$. When $\alpha\geqq\beta$ and $\mathfrak{B}=\mathbf{B}(\beta)$, we see that $\gamma$ is a subnet of $\alpha$ such that $\gamma\sim\beta$. If we permit ourself the luxury of identifying equivalent nets, we may regard the expressions "$\alpha\geqq\beta$" and "$\beta$ is a subnet of $\alpha$" as synonymous. In the course of the preparation of this note, we have become aware (through conversation) of an unpublished theory of set-nets developed by B. J. Pettis. Pettis gives a generalization of the notion of subnet which is equivalent for point-nets (that is, for nets) to the relation $\alpha\geqq\beta$ of the present note. The discussion of "ultimate" concepts in the present notation is left to the interested reader.

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## MATHEMATICAL NOTES

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### CONVERGENCE OF SERIES WITH POSITIVE TERMS*

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**1. Introduction.** It is the purpose in what follows to prove necessary and sufficient conditions for convergence of series with positive terms that serve as a general framework for short proofs of the sufficient conditions of many of the known tests for convergence or divergence of such series (see [**1**], [**2**], [**3**]).

It will be understood that each series considered will be a series with positive terms unless the contrary is stated.

**2. Necessary and sufficient conditions.** As the basis of the conditions to be proved, we suppose as known the general criterion that a series with positive

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terms is convergent if and only if its partial sums are bounded.

THEOREM 1. *A necessary and sufficient condition that a series* $\sum a_n$ *with positive terms converge is that there exist positive numbers* $p_n$ *and a nonnegative integer* $k$ *such that* $p_n - p_{n+1} \geq a_{k+n}$ *for each* $n = 1, 2, \cdots$.

*Proof.* The condition is necessary since, for $k=0$, we may take $p_n = \sum_{i=n}^{\infty} a_i$. On the other hand, the condition implies $p_1 > p_1 - p_{n+1} \geq \sum_{i=1}^{n} a_{k+i}$. Hence the partial sums of the series are bounded above by $p_1 + s_k$, where $s_k$ is the $k$th partial sum. Therefore the series is convergent by the general criterion and the condition is sufficient.

The condition is necessary but its use in practice will be due to its sufficiency. For this reason, the following theorem is included for use in tests for divergence.

THEOREM 2. *A necessary and sufficient condition that a series* $\sum a_n$ *with positive terms diverge is that there exist an unbounded set of positive numbers* $p_n$ *and a nonnegative integer* $k$ *such that* $0 < p_{n+1} - p_n \leq a_{k+n}$ *for each* $n = 1, 2, \cdots$.

*Proof.* The condition is necessary since, for $k=1$, we may take $p_n = \sum_{i=1}^{n} a_i$. On the other hand, the condition implies $p_{n+1} - p_1 \leq \sum_{i=1}^{n} a_{k+i}$. Hence the partial sums $s_{k+n}$ are not bounded above since the set of numbers $p_{n+1}$ are increasing and unbounded. Therefore the series is divergent by the general criterion and the condition is sufficient.

**3. Tests for convergence and divergence.** In order to set up a test for convergence, we only need to specify a nonnegative integer $k$ and a set of positive numbers $p_n$ and require that the inequality in Theorem 1 be satisfied. For example, if $\sum c_n$ is a convergent series, a set of positive numbers is defined by the equation $p_n = \sum_{i=n}^{\infty} c_i$ and $p_n - p_{n+1} = c_n$. Hence, for $k=0$, the inequality of Theorem 1 requires that $c_n \geq a_n$, which is the familiar comparison test. Similar remarks hold for tests for divergence based on Theorem 2.

Because of the principle explained in the preceding paragraph, we will list the choices of $k$ and $p_n$, which, when associated with the inequality in Theorem 1 or Theorem 2, immediately give many of the familiar tests. The tests for convergence below have reference to Theorem 1 while those for divergence have reference to Theorem 2.

*Comparison test.*
*Convergence*: $p_n = \sum_{i=n}^{\infty} c_i$, $k=0$; $\sum c_n$ convergent.
*Divergence*: $p_n = \sum_{i=1}^{n} d_i$, $k=1$; $\sum d_n$ divergent.

*Root test.*
*Convergence*: $p_n = \theta^n (1-\theta)^{-1}$, $0 < \theta < 1$, $k=0$.
*Divergence*: $p_n = n$, $k=0$.

*Ratio comparison test.*
*Convergence*: $p_n = (a_1/c_1) \sum_{i=n}^{\infty} c_i$, $k=0$; $\sum c_n$ convergent.
*Divergence*: $p_n = (a_1/d_1) \sum_{i=1}^{n} d_i$, $k=1$; $\sum d_n$ divergent.

*Remark* 1. The last test will appear in its familiar form when we observe that, for convergence, the condition is satisfied if we require $a_{n+1}/a_n \leq c_{n+1}/c_n$; while, for divergence, the condition is satisfied if we require $a_{n+1}/a_n \geq d_{n+1}/d_n$.

*Ratio test.*
*Convergence*: $p_n = a_n/\theta$, $\theta > 0$, $k = 1$.
*Divergence*: $p_n = na_1$, $k = 1$.

*Remark* 2. For divergence, the condition is satisfied if $a_n \leq a_{n+1}$.

*Raabe's test.*
*Convergence*: $p_n = na_n/\theta$, $\theta > 0$, $k = 1$.
*Divergence*: $p_n = a_1 \sum_{i=1}^{n} 1/i$, $k = 1$.

*Remark* 3. For divergence, the condition is satisfied if $na_n \leq (n+1)a_{n+1}$. Here the divergence of the harmonic series has been assumed. Later it will be shown to diverge as a consequence of Theorem 2.

*Integral test.*
*Convergence*: $p_n = \int_n^\infty f(x)dx$, $f(n) = a_n$, $k = 1$.
*Divergence*: $p_n = \int_1^n f(x)dx$, $f(n) = a_n$, $k = 0$.

*Remark* 4. For convergence, it is assumed that the improper integral converges; for divergence, that it diverges. Each condition is satisfied if $f(x)$ is monotonely decreasing. The test, as given, is more general than Cauchy's integral test since it does not require that $a_n \geq a_{n+1}$ nor that $f(x)$ decrease monotonely.

*Kummer's test.*
*Convergence*: $p_n = d_n a_n/\theta$, $\theta > 0$, $d_n > 0$, $k = 1$.
*Divergence*: $p_n = a_1 d_1 D_n$, $d_1 > 0$, $D_n > 0$, $k = 1$.

*Remark* 5. It is easy to see that these lead to necessary and sufficient conditions, since they are essentially a restatement of Theorems 1 and 2 with a change in notation for the set of positive numbers $p_n$. The necessity of Kummer's conditions seems not to have been known previously (see [**1**], [**2**], [**3**]). For divergence, the condition is satisfied if we require the divergence of $\sum 1/d_n$ and that $a_n d_n \leq a_{n+1} d_{n+1}$, since then there exists an unbounded set $D_n$ such that $0 < D_{n+1} - D_n < 1/d_{n+1}$.

*Abel-Dini-Pringsheim test.* Let $\sum d_n$ be a divergent series and $D_n$ be its $n$th partial sum. Let $\theta = 1 + p > 1 + 1/m$, where $m$ is a positive integer. Then the choice $p_n = m/D_{n-1}^{1/m}$ and the inequalities

$$m/D_{n-1}^{1/m} - m/D_n^{1/m} \geq D_n - D_{n-1}/D_n D_{n-1}^{1/m} \geq d_n/D_n D_{n-1}^{p} \geq d_n/D_n^{\theta},$$

together with Theorem 1, prove the convergence of the series $\sum d_n/D_n^\theta$, $\theta > 1$, as well as a series with any one of the terms in the inequalities as general term.

Similarly, the choice $p_n = D_n^{1-\theta}, \theta \leq 1$ leads to the divergence of the series $\sum d_n/D_n^\theta$.

*Remark* 6. Cauchy's condensation test and Ermakoff's test can be derived as special cases of those already given. It should be clear now that the list of tests can be enlarged with ease by the proper choice of the numbers $p_n$. Rather than this, we will develop the theory along another line in the next section.

**4. Convergent and divergent series.** The following theorem is useful to obtain specific convergent series.

THEOREM 3. *If $f(x)$ and $-f'(x)$ are positive functions and $-f'(x)$ is monotonely decreasing for $x \geq 1$, then the series $\sum -f'(n)$ converges.*

*Proof.* The hypothesis and the mean value theorem imply $f(x) - f(x+1) = -f'(x+\theta) \geq -f'(x+1)$, where $0 < \theta < 1$. Hence it follows from Theorem 1 that the series $\sum -f'(n+1)$ (and thus also $\sum -f'(n)$) converges.

Note that $-f'(x)$ is monotonely decreasing provided $f''(x) > 0$. This immediately implies the convergence of the following series:

$$\sum br^n, \quad 0 < r < 1; \quad f(x) = ar^x, \quad b = -a \log r,$$
$$\sum bn^{-1-\theta}, \quad \theta > 0; \quad f(x) = ax^{-\theta}, \quad b = a\theta,$$
$$\sum b(n \log n \cdots \log_{p-1} n)^{-1}(\log_p n)^{-1-\theta}, \quad \theta > 0, \quad f(x) = a(\log_p x)^{-\theta},$$

where $\log_p n$ denotes $\log \log \cdots \log n$ with $p$ operations and $b = a\theta$. This list could be enlarged without difficulty.

THEOREM 4. *If $f(x)$ is a positive unbounded function and $f'(x)$ is a positive, monotonely decreasing function for $x \geq 1$, then the series $\sum f'(n)$ diverges.*

The proof follows immediately from the mean value theorem and Theorem 2. From this theorem, it is seen that the following series diverge:

$$\sum br^n, \quad r > 1; \quad f(x) = ar^x, \quad b = a \log r \neq 0,$$
$$\sum bn^{-1+\theta}, \quad 0 < \theta \leq 1; \quad f(x) = ax^\theta, \quad b = a\theta \neq 0,$$
$$\sum b(n \log n \cdots \log_{p-1} n)^{-1}(\log_p n)^{-1+\theta}, \quad 0 < \theta \leq 1, \quad f(x) = a(\log_p n)^\theta,$$

where $b = a\theta \neq 0$. Note that the last case includes the divergence of the harmonic series for $a = \theta = p = 1$.

The last two theorems may be restated so as to be used directly as tests. This is done in the next theorem for Theorem 3.

THEOREM 5. *If an indefinite integral of $f(x)$ is a negative function and $f(x)$ is a positive, monotonely decreasing, and continuous function for $x \geq 1$, then the series $\sum f(n)$ converges.*

It is interesting to observe that the last theorem is an "integral test" without a direct reference to an improper integral.

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## POLYNOMIALS WITH THE BINOMIAL PROPERTY

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Let us call $\{b_n(x)=\sum_{i=0}^{n} b_{ni}x^i,\ b_{nn}\neq 0\}$ a binomial set of polynomials if the relation

$$b_n(x+y)=\sum_{i=0}^{n}\binom{n}{i}b_i(x)b_{n-i}(y),\qquad n=0,1,\cdots,\tag{1}$$

is satisfied. Two examples of binomial sets are the powers, $\{x^n\}$, and the factorial polynomials $\{x^{(n)}=x(x-1)\cdots(x-n+1)\}$. It seems worth while to point out that binomial sets are (except for constant multipliers) the basic sets of polynomials of type zero introduced by I. M. Sheffer.* His basic polynomials are defined by his theorem:

*Let $J(B)=c_1B'+c_2B''+\cdots$ ($c_n$=constant, $c_1\neq 0$). To each operator $J$ there corresponds one and only one polynomial set $\{B_n(x)\}$ (which we call the basic set) such that*

$$B_0(x)=1,\quad B_n(0)=0,\quad n>0,\qquad J(B_n(x))=B_{n-1}(x).$$

The assumption, $c_n$=constant, classifies $J$ as a type zero operator. For the case where the operator is $\sin D$,

$$J(B)=B'-\frac{1}{3!}B'''+\frac{1}{5!}B^{v}-\cdots,$$

and the polynomial sets are

*binomial set*: $1,\ x,\ x^2,\ x^3+x\quad x^4+4x^2,\ x^5+10x^3+9x,\cdots,$

*basic set*: $1,\ x,\ \frac{1}{2!}x^2,\ \frac{1}{3!}(x^3+x),\ \frac{1}{4!}(x^4+4x^2),\ \frac{1}{5!}(x^5+10x^3+9x),\cdots.$

To show the correspondence between basic and binomial sets, we start with

THEOREM 1. *If $\{B_n(x)\}$ is a type zero basic set, then $\{n!B_n(x)\}$ is a binomial set.*

This result follows at once from Sheffer's relation $e^{xH(t)}=\sum_{i=0}^{\infty}B_i(x)t^i$ by equating powers of $t$ in

$$e^{xH(t)}e^{yH(t)}=e^{(x+y)H(t)}=\sum_{n=0}^{\infty}t^n\sum_{i=0}^{n}B_i(x)B_{n-i}(y)=\sum_{n=0}^{\infty}B_n(x+y)t^n.$$

Before proceeding to the converse theorem, we note

THEOREM 2. *If $\{b_n(x)\}$ is a binomial set, then $b_0(x)=1$, $b_n(0)=0$, $n>0$.*

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* I. M. Sheffer, Some properties of polynomial sets of type zero. Duke Math. J., vol. 5, 1939.

The first two equations of (1) are

$$b_0(x+y) = b_0(x)b_0(y), \qquad b_1(x+y) = b_0(x)b_1(y) + b_1(x)b_0(y).$$

Thus the constant $b_0(x)=1$. Setting $x=y=0$ in $b_1(x+y)$, we get $b_1(0)=0$ and an induction gives $b_n(0)=0$, $n>0$.

THEOREM 3 (*converse of Theorem* 1). *If* $\{b_n(x)\}$ *is a binomial set, there exists a type zero operator* $J$ *whose basic set is* $\{(1/n!)b_n(x)\}$.

Having Theorem 2, it suffices to produce an operator $J$ such that

$$J(b_n(x)) = nb_{n-1}(x), \qquad J(y) = c_1y' + c_2y'' + \cdots \ (c_1 \neq 0).$$

Simple algebraic manipulation will procure a few terms of $J$. If $b_0(x)=1$, $b_1(x)=ax$, $b_2(x)=a^2x^2+bx, \cdots$, the operator is $J(y)=(1/a)y'-(b/2a^3)y''+\cdots$. A step-by-step process will produce the remaining terms uniquely. Suppose that the operator $J_{n-1}$ has the properties

$$J_{n-1}(b_k(x)) = kb_{k-1}(x), \qquad k = 1, \cdots, n-1,$$
$$J_{n-1}(y) = \frac{1}{a}y' + \cdots + c_{n-1}y^{(n-1)}.$$

This operator possesses a unique basic set

$$b_0(x), \cdots, \frac{1}{n-1!}b_{n-1}(x), \frac{1}{n!}b_n^*(x), \cdots,$$

whose first $n$ terms coincide with the given binomial set (except for the constant multipliers). From Theorem 1, $b_0(x), \cdots, b_{n-1}(x), b_n^*(x), \cdots$, must also form a binomial set. A relation connecting $b_n(x)$ and $b_n^*(x)$ can be obtained from (1):

$$\sum_{i=1}^{n-1}\binom{n}{i}b_i(x)b_{n-i}(y) = b_n(x+y) - b_n(x) - b_n(y) = b_n^*(x+y) - b_n^*(x) - b_n^*(y).$$

If $b_n(x) = \sum_{i=1}^{n} r_ix^i$, the expansion of the above expression is

$$b_n(x+y) - b_n(x) - b_n(y) = [nr_nx^{n-1} + \cdots + 2r_2x]y + \text{terms in } y^2.$$

Since the expression obtained from $b_n^*(x)$ must be identical with this, the two $n$th degree polynomials can differ only in their coefficients of $x$, *i.e.*, $b_n(x)=b_n^*(x)+cx$, $c=r_1-r_1^*$. Let $J_n=J_{n-1}-(c/ar_nn!)D^n$. Then

$$J_n(b_n(x)) = J_{n-1}(b_n^*(x)) + J_{n-1}(cx) - (c/ar_nn!)D^n[b_n^*(x) + cx]$$
$$= nb_{n-1}(x) + (c/a) - (c/ar_nn!)n!r_n = nb_{n-1}(x).$$

Thus $(1/n!)b_n(x)$ is a basic polynomial of the operator $J_n$, and $\{(1/n!)b_n(x)\}$ is the basic set of the operator $J \equiv J_\infty$.

## SOME INEQUALITIES INVOLVING HERMITE POLYNOMIALS

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Problem 4215 [1946, 470], this MONTHLY, indicates that

$$\Delta_n(x) = H_n^2(x) - H_{n+1}(x)H_{n-1}(x) = (n-1)!\sum_{i=0}^{n-1} H_i^2(x)/i!,$$

where $H_n(x) = (-1)^n e^{x^2/2} d^n(e^{-x^2/2})/dx^n$ is the Hermite polynomial of degree $n$. From this we obtain immediately the inequality

$$\Delta_n(x) > 0, \text{ all } x,\ n \geq 1. \tag{1}$$

Mukherjee and Nanjundiah in [1] establish the identity $n\Delta_n(x) = [H_n'(x)]^2 - H_n(x)H_n''(x)$, with which (1) can also be proved. Sharper estimates of (1) may be found in [2] and [3]. The corresponding inequality with the derivatives of Hermite polynomials is treated in [4] and [5].

Toscano in [6] establishes the identity

$$\delta_n(x) = H_{n+1}(x)H_{n+2}(x) - H_n(x)H_{n+3}(x) = 2xn!\sum_{i=0}^{[n/2]} H_{n-2i}^2(x)/(n-2i)!$$

from which follows

$$\delta_n(x)\begin{Bmatrix} >0, & x > 0 \\ =0, & x = 0 \\ <0, & x < 0 \end{Bmatrix}, \qquad n \geq 0. \tag{2}$$

Toscano also shows that $\Delta_n'(x) = (n-1)\delta_{n-2}(x)$. Hence

$$\Delta_n'(x)\begin{Bmatrix} >0, & x > 0 \\ =0, & x = 0 \\ <0, & x < 0 \end{Bmatrix}, \qquad n \geq 2. \tag{3}$$

In this paper we establish similar inequalities. First, (1) has two immediate generalizations of a different nature:

$$[H_n(x)]^{2(2r+1)} - [H_{n+1}(x)]^{2r+1}[H_{n-1}(x)]^{2r+1} > 0, \tag{4}$$

$$r \text{ a positive integer, all } x,\ n \geq 1.$$

Simple examples indicate that if $2r+1$ is replaced by an even integer, the inequality is no longer valid.

$$H_n^2(x) - kH_{n+1}(x)H_{n-1}(x) > 0, \quad 0 \leq k \leq 1, \quad \text{all } x,\ n \geq 1. \tag{5}$$

One can show that for $k$ outside this range, the expression on the left changes sign.

Next we show that

(6) $(n+1)H_n^2(x) - (n-1)H_{n+2}(x)H_{n-2}(x) \geq 0$, all $x$, $n \geq 2$, with equality for $x = 0$ only.

Using the recurrence relation

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x), \tag{7}$$

the left-hand side of (6) can be written as $x\delta_{n-1}(x)$, and the result follows from (2). $H_2^2(x) - H_0(x)H_4(x) = 4x^2 - 2$ shows that $n+1$ and $n-1$ cannot be replaced by unity in (6).

We now prove

$$H_n(x)H_n'(x) - H_{n+1}(x)H_{n-1}'(x) \begin{Bmatrix} >0, & x > 0 \\ =0, & x = 0 \\ <0, & x < 0 \end{Bmatrix}, \qquad n \geq 1. \tag{8}$$

Using the relation

$$H_n'(x) = nH_{n-1}(x), \tag{9}$$

the left-hand side of (8) becomes $nH_n(x)H_{n-1}(x) - (n-1)H_{n+1}(x)H_{n-2}(x)$. Using (7) to replace $H_{n-1}(x)$ and $H_{n-2}(x)$ in this expression, yields $H_n(x)H_n'(x) - H_{n+1}(x)H_{n-1}'(x) = x\Delta_n(x)$, and the result follows from (1).

Next we show that

$$[H_n'(x)]^2 - H_{n+1}''(x)H_{n-1}(x) \leq 0, \text{ all } x,\ n \geq 1. \tag{10}$$

With the use of (9) and the differential equation

$$H_n''(x) - xH_n'(x) + nH_n(x) = 0, \tag{11}$$

we obtain

$$[H_n'(x)]^2 - H_{n+1}''(x)H_{n-1}(x) = n^2H_{n-1}^2(x) - H_{n-1}(x)[xH_{n+1}'(x) - (n+1)H_{n+1}(x)].$$

The use of (7) and (9) to eliminate $H_{n+1}(x)$ and $H_{n+1}'(x)$ respectively then yields $[H_n'(x)]^2 - H_{n+1}''(x)H_{n-1}(x) = -nH_{n-1}^2(x)$, from which the result follows.

Now, we show

(12) $[H_n'(x)]^2 - H_{n+1}(x)H_{n-1}''(x) \geq 0$, all $x$, $n \geq 1$, equality for $x = 0$ and $n$ odd only.

Multiplying by $n$ both sides of the inequality in (6), we obtain

$$n(n+1)H_n^2(x) - n(n-1)H_{n+2}(x)H_{n-2}(x) \geq 0.$$

Then

$$(n+1)H_n^2(x) + n(n+1)H_n^2(x) - n(n-1)H_{n+2}(x)H_{n-2}(x) \geq 0,$$

or

$$(n+1)^2H_n^2(x) - n(n-1)H_{n+2}(x)H_{n-2}(x) \geq 0,$$

with equality for $x=0$ and $n$ odd only. Using (9) leads to the result.

Next, we have

(13) $[H_n''(x)]^2 - H_n(x)H_n^{(\mathrm{iv})}(x) \geq 0$, all $x$, $n \geq 2$, with equality for $x = 0$ and $n$ odd only.

With the use of (9), the left-hand side of (13) becomes $n^2[H_{n-1}'(x)]^2 - n(n-1)H_n(x)H_{n-2}''(x)$ and the result follows by (12) since $n(n-1)/n^2<1$.

Next we establish the identity

$$n\Delta_n''(x) = [H_n''(x)]^2 - H_n(x)H_n^{(\mathrm{iv})}(x). \tag{14}$$

With the use of (7) and (9), we have

$$n\Delta_n(x) = nH_n^2(x) - xH_n(x)H_n'(x) + [H_n'(x)]^2.$$

Using (11), we can write

$$n\Delta_n'(x) = x[H_n'(x)]^2 - H_n(x)H_n'(x) - xH_n(x)H_n''(x).$$

Then $n\Delta_n''(x)=xH_n'(x)H_n''(x)-2H_n(x)H_n''(x)-xH_n(x)H_n'''(x)$. From (11) we obtain $H_n^{(\mathrm{iv})}(x)=xH_n'''(x)+(2-n)H_n''(x)$, which with the above yields the result.

(15) $\Delta_n''(x) \geq 0$, all $x$, $n \geq 2$, with equality for $x = 0$ and $n$ odd only.

In general, the derivatives of $\Delta_n(x)$ of order greater than 2 do not remain of one sign in any half line.

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## A GENERAL FORMULA FOR CIRCULAR PERMUTATIONS

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The general formula for circular permutations does not seem to have been discussed in textbooks. In this note, we establish a general formula for the number of circular permutations of a set of $n$ objects containing similar elements. We first prove the following:

THEOREM 1. *Let the set of $n$ objects contain $r_1$ elements $a_1$, $r_2$ elements $a_2, \cdots, r_p$ elements $a_p$. Let $h$ be the highest common factor of $r_\nu$ $(\nu=1, \cdots, p)$. Then, if $h$ is a prime, the number $P$ of circular permutations of the $n$ objects is given by*

$$P = (1/n)\left\{\frac{n!}{r_1! \cdots r_p!} - \frac{(n/h)!}{(r_1/h)! \cdots (r_p/h)!}\right\} + \frac{((n/h)-1)!}{(r_1/h)! \cdots (r_p/h)!}. \tag{1}$$

*Proof.* Since $h$ is a prime, we can divide the $n$ objects into $h$ similar sections, each containing $r_1/h$ elements $a_1, \cdots, r_p/h$ elements $a_p$. If each section is denoted by $[C]$, the $n$ objects may be arranged as

$$[C] + [C] + \cdots + [C] \qquad \text{(to } h \text{ terms)}. \tag{2}$$

Take, for instance, a special permutation $[C_\mu]$ formed by permuting the elements of $[C]$. If we arrange the given $n$ objects as $[C_\mu]+ \cdots +[C_\mu]$ (to $h$ terms), then this is clearly one of the permutations of the $n$ objects. This permutation, as a circular permutation, is unchanged if we shift the first element to the end, the second to the end, and so on, and at last, the $n/h$th element to the end. Thus, the linear permutations of the $n$ objects formed by linking $h$ similar sections of $[C]$ is in an $n/h$: 1 correspondence with the circular permutations of the $n$ objects formed by the same method of linkage. Since $r_1/h, \cdots, r_p/h$ have no common factor, we cannot divide into further sub-sections of elements. Now, the number of linear permutations obtained by linking $h$ of the sections $[C]$ of the same form of permutation is equivalent to the number of the linear permutations formed by permuting the elements of $[C]$. If $k$ denotes the number of linear permutations of each $[C]$, it is known that

$$k = \frac{(n/h)!}{(r_1/h)! \cdots (r_p/h)!}. \tag{3}$$

It is also known that the number $K$ of linear permutations of the $n$ objects is given by $K=n!/(r_1! \cdots r_p!)$. It is evident that there is an $n$: 1 correspondence between the linear permutations and the circular permutations of the $n$ objects, except for the above permutations formed by linking $h$ similar sections $[C]$. Therefore, it follows that $P=(1/n)(K-k)+k/(n/h)$, and this is the result given in (1).

We obtain the following corollary by taking $h=1$ in Theorem 1, since the number of circular permutations cannot be a fraction.

COROLLARY. *If $r_1, \cdots, r_p$ have no common factor, then $(n-1)!/(r_1! \cdots r_p!)$ is an integer.*

By means of Theorem 1 we can prove the following:

THEOREM 2. *In the notation of Theorem 1, let $h=h_0h_1 \cdots h_m$, where $h_0=1$ and $h_1, \cdots, h_m$ are primes, and let $H_i=h_0h_1 \cdots h_i$. Then*

$$
\begin{aligned}
P = \sum_{i=0}^{m-1} \frac{1}{n/H_i}\left\{\frac{(n/H_i)!}{(r_1/H_i)! \cdots (r_p/H_i)!} - \frac{(n/H_{i+1})!}{(r_1/H_{i+1})! \cdots (r_p/H_{i+1})!}\right\} \\
+ \frac{((n/h)-1)!}{(r_1/h)! \cdots (r_p/h)!}.
\end{aligned}
\tag{4}
$$

*Proof.* We divide the $n$ objects into $h_1$ similar sections $[C_1]$, each containing $r_1/H_1$ elements $a_1, \cdots, r_p/H_1$ elements $a_p$. Next, we divide $[C_1]$ into $h_2$ similar sections $[C_2]$, each containing $r_1/H_2$ elements $a_1, \cdots, r_p/H_2$ elements $a_p$. By successive similar operations, we arrive at $[C_m]$, which contains $r_1/H_m$ elements $a_1, \cdots, r_p/H_m$ elements $a_p$. From the given conditions it is seen that $r_1/H_m, \cdots, r_p/H_m$ have no common factor. By Theorem 1, we know that the relation between the linear permutations formed by linking $h_1$ similar sections $[C_1]$ and the circular permutations for the same arrangement is an $n/H_1$: 1 correspondence, the relation between the linear permutations obtained by linking $h_2$ similar sections $[C_2]$ and the circular permutations for the same arrangement is an $n/H_2$: 1 correspondence, and so on. If we write

$$k_i = \frac{(n/H_i)!}{(r_1/H_i)! \cdots (r_p/H_i)!} \qquad (i = 0, \cdots, m),$$

then, from the arguments of Theorem 1, it is easily seen that

$$P = \sum_{i=0}^{m-1} \frac{1}{n/H_i}(k_i - k_{i+1}) + \frac{k_m}{n/h},$$

and this is the result given in (4).

## BEST FITTING INTEGRAL CURVES OF LINEAR DIFFERENTIAL EQUATIONS

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1. We are given a set of points $(x_k, \bar{y}_k)$ $k=0, \cdots, m$. These points are number pairs, the numbers representing measurements obtained by physical experiment. They differ from a set of points $(x_k, y_k)$ which lie on an integral curve of some linear differential equation of known order and known driving function by the random error usually present in any set of physical measurements. We wish to determine the coefficients of the differential equation together with a set of initial conditions so that the resulting integral curve will be a best fitting

curve for the given set of data. The term "best fitting curve" will be defined after some preliminary theory has been developed. We proceed to a more special problem.

**2.** Given a differential equation

$$b_0 y + b_1 y' + \cdots + b_m y^{(m)} = D(x), \tag{1}$$

where $D(x)$ is a function such that

$$\overline{M}_k = \int_0^\infty x^k D(x)dx$$

are finite constants not all 0 for all $k=0, \cdots, 2m$ and the $b_k$ are disposable parameters. Let $y=F(x)$ be a function with this same property that

$$M_k = \int_0^\infty x^k F(x)dx$$

exist for $k=0, \cdots, 2m$.

THEOREM. *If there exist a set of numbers $b_k$ such that $y=F(x)$ is a solution of* (1), *then this set is uniquely determined by $M_k$ and $\overline{M}_k$ provided the determinant $|C_{i,j}| \neq 0$ where $i=1, \cdots, m+1, j=1, \cdots, m+1$, and $C_{i,j}=C_k=(-1)^k M_k/k!$, $k=i+j-2$.*

*Proof*. Let the Laplace transform of $F(x)$ be $f(s)=\mathcal{L}[F(x)]=\int_0^\infty F(x)e^{-sx}dx$. Differentiating gives the expressions

$$f^{(k)}(0) = (-1)^k M_k \tag{2}$$

for the first $2m$ derivatives of $f(s)$ at $s=0$, their existence being assured by the assumption that $M_k$ exist. Similarly if $g(s)$ is the Laplace transform of $D(x)$, then $g^{(k)}(0)=(-1)^k\overline{M}_k$, $k=0, \cdots, 2m$. Then

$$f(s) = \frac{g(s) + a_0 + a_1 s + \cdots + a_{m-1}s^{m-1}}{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}, \tag{3}$$

where

$$a_k = \sum_{k+1}^{m} b_i F^{i-k-1}(0). \tag{4}$$

Equations (2) and (3) will now be used to obtain $2m+1$ equations for determining the $2m+1$ parameters $a_k$ and $b_k$. The equations are all linear in $a_k$ and $b_k$ and may be found by the following device.

Expand $f(s)$ and $g(s)$ by MacLaurin's series. Then the coefficients $c_k$ and $d_k$, respectively, are given by

(5) $$c_k = f^{(k)}(0)/k! = (-1)^k M_k/k!, \qquad d_k = (-1)^k \overline{M}_k/k!.$$

Multiplying both sides of equations (3) by $b_0+b_1s+\cdots+b_ms^m$ and equating the coefficients of like powers of $s$ gives

(6) $$a_k = c_0b_k + c_1b_{k-1} + \cdots + c_kb_0 - d_k, \qquad k = 0, \cdots, m-1;$$

(7) $$c_kb_m + c_{k+1}b_{m-1} + \cdots + c_{k+m}b_0 = d_{k+m}, \qquad k = 0, \cdots, m.$$

Under the conditions of the theorem equations (6) and (7) have a unique solution and the theorem is proved.

The homogeneous case can be solved by setting $D(x)\equiv 0$. If $|C_{i,j}|\neq 0$, then each $b_k=0$ by the previous theorem and there exists no homogeneous differential equation which has $y=F(x)$ as its solution. However, if the rank of the matrix $\|C_{ij}\|$ is $m$ and $y=F(x)$ satisfies such a differential equation it is unique since equations (7) now determine uniquely the ratios of the $b_k$'s.

**3.** The theorem of the preceding paragraph is true under much weaker conditions on $F(x)$ and $D(x)$. We demand only that there exist a $p\geqq 0$ such that

$$M_k = \int_0^\infty x^kF(x)e^{-px}dx \quad \text{and} \quad \overline{M}_k = \int_0^\infty x^kD(x)e^{-px}dx$$

exist for $k=0, \cdots, 2m$. For if we transform (1) by the transformation $Z=e^{-px}y$, the new equation becomes

(8) $$B_0Z + B_1Z' + B_2Z'' + \cdots + B_mZ^{(m)} = e^{-px}D(x).$$

If $y=F(x)$ is a solution of (1), then $Z=e^{-px}F(x)$ is a solution of (8). By the above theorem the $B_k$ are determined uniquely and the inverse transformation will produce equation (1) uniquely.

**4.** *Definition: Let $y=G(x)$ be a curve such that*

$$\mu_k = \int_0^\infty x^kG(x)e^{-px}dx$$

*exist for some $p\geqq 0$ and for $k=0, \cdots, 2m$. A curve $y=F(x)$ is a best fitting integral curve of a linear differential equation* (1) *with respect to $y=G(x)$ and for a given $p$ if it satisfies* (1) *and if*

$$M_k = \int_0^\infty x^kF(x)e^{-px}dx$$

*exist and are equal to $\mu_k$.*

We return to the problem of the first paragraph. Let $y=G(x)$ be an arbitrary curve containing the point set $(x_k, \bar{y}_k)$. Compute the moments $\mu_k$ and then compute $d_k$ and $c_k$, replacing $M_k$ by $\mu_k$. Equation (7) is solved for the values $b_k$. Equations (6) and then (4) are used to compute the initial conditions. An analogue computer may now be used to draw $y=F(x)$, or equation (1) with computed initial conditions may be used to solve for $y=F(x)$.

If $y=G(x)$ does not approximate an integral curve of (1), the best fitting integral curve may be of little value. However, the method may be very useful for fitting curves which approach the $x$-axis asymptotically where polynomial curves are impractical. Many such curves approximate integral curves of linear homogeneous differential equations. When applicable, the method provides an excellent means of preparing tabular data for use in an analogue computer.

## A SUBSET OF THE COUNTABLE ORDINALS

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Let $\Lambda$ be the space of all countable ordinals with the order topology. There are two obvious types of uncountable subsets of $\Lambda$:

(1) *sets which contain some uncountable closed set,*
(2) *sets which are contained in the complement of some uncountable closed set.*

The purpose of this paper is to show that there is a third type of uncountable subset of $\Lambda$:

(3) *sets which intersect every uncountable closed set but which contain no uncountable closed set.*

This was surprising to the author primarily because of the following fact:

(A) *the intersection of any countable family of sets of type* (1) *is again a set of type* (1).

*Proof of the existence of sets of type* (3). Let $f$ be a one-to-one transformation of $\Lambda$ onto a subset of a line $L$. For each positive integer $n$, let $X_n$ be a countable collection of intervals covering $L$ of length less than $1/n$ such that, for $n>1$, $X_n$ is a refinement of $X_{n-1}$. For $x$ in $X_n$, let $\lambda(x)$ be the set of all $\alpha$ in $\Lambda$ such that $f(\alpha)$ is in $x$. We will show that, for some $x$, $\lambda(x)$ is of type (3).

Let us assume that there is no $n$ and $x$ in $X_n$ such that $\lambda(x)$ is of type (3). We will now show that there are intervals $x_1 \supset x_2 \supset x_3 \supset \cdots$ such that, for each $n$, $x_n$ belongs to $X_n$ and $\lambda(x_n)$ is of type (1). By (A), $\bigcap_{n=1}^{\infty} \lambda(x_n)$ is uncountable, but this is impossible since $\bigcap_{n=1}^{\infty} x_n$ is a single point and $f$ is one-to-one.

The complement of every countable set and of every set of type (2) is of type (1). Therefore, by (A), at least one term $x_1$ of $X_1$ is such that $\lambda(x_1)$ is of type (1). Similarly, if $x_n$ has been defined as a term of $X_n$ such that $\lambda(x_n)$ is of type (1), then by the same argument there is a subinterval $x_{n+1}$ of $x_n$ belonging to $X_{n+1}$ such that $\lambda(x_{n+1})$ is of type (1).

## GENERALIZATIONS OF THE THEOREM OF CHASLES

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**1. Introduction.** We are concerned here with generalizations of the following classical result due to Chasles [4]:

*If the vertices of one triangle are the poles of the sides of another triangle relative to a symmetric plane polarity, and, if the lines joining corresponding pairs of vertices of the two triangles are well defined, then these lines are concurrent.*

Using the language of inner product spaces, it is possible to state and prove these results in a way that does not depend essentially on the dimension of the space involved. The principal tool used is the Parseval identity relative to a complete biorthogonal set of vectors. In the next section, we look at the finite-dimensional case (which has previously been considered in [1], [2], [3], [4]); in Section 3 we remove this restriction.

**2. The finite case.** Throughout this section, we assume that we have an $n$-dimensional vector space $V$ over an arbitrary field together with a scalar-valued, bilinear, symmetric, non-degenerate inner product $(x, y)$ defined for pairs of vectors $x, y$ in $V$. Consider now the following theorem (*cf.* [1], [2], [3]):

THEOREM. *Assume given $2n$ vectors $u_i$, $v_i$, $i=1, \cdots, n$, and a plane* $\Pi$ *such that* (1) *the u's and v's form a biorthogonal set, $(u_i, v_j)=\delta_{ij}$,* (2) *the vectors $u_i$ and $v_i$ are linearly independent for each $i=1, \cdots, n$, and* (3) *the plane* $\Pi$ *contains non-zero vectors orthogonal to the planes $\Pi_i=\{u_i, v_i\}$, $i=2, \cdots, n$. Then the plane* $\Pi$ *also contains a non-zero vector orthogonal to $\Pi_1=\{u_1, v_1\}$.*

We first remark that the theorem of Chasles, restated, is the case $n=3$. Next, we note that each condition of (3) can be stated in several equivalent ways: $\Pi_i$ contains a vector orthogonal to $\Pi$, or, the orthogonal complement of $\Pi$ contains a non-zero vector in common with $\Pi_i$. The next lemma gives yet another such condition which we use in the proof of the theorem.

LEMMA. *If $x$ and $y$ form a basis for the plane* $\Pi$, *then the plane* $\Pi$ *contains a non-zero vector orthogonal to $\Pi_i$ if and only if $(x, u_i)(y, v_i)-(x, v_i)(y, u_i)=0$.*

*Proof.* The non-zero vector $\alpha x+\beta y$ of $\Pi$ is orthogonal to $\Pi_i$ if and only if $(\alpha x+\beta y, u_i)=0$ and $(\alpha x+\beta y, v_i)=0$; *i.e.* $\alpha(x, u_i)+\beta(y, u_i)=0$ and $\alpha(x, v_i)+\beta(y, v_i)=0$. These last equations have a non-zero solution for $\alpha, \beta$ if and only if $(x, u_i)(y, v_i)-(x, v_i)(y, u_i)=0$.

*Proof of Theorem.* By the Parseval identity we have

$$(*) \qquad (x, y) = \sum_{i=1}^{n} (x, u_i)(y, v_i).$$

Interchanging $x$ and $y$, we get $(y, x)=\sum_{i=1}^{n} (y, u_i)(x, v_i)$. Since $(x, y)=(y, x)$, this gives

$$\sum_{i=1}^{n} [(x, u_i)(y, v_i) - (x, v_i)(y, u_i)] = 0.$$

By the hypotheses (3), and, by the lemma, $(x, u_i)(y, v_i) - (x, v_i)(y, u_i) = 0$, $i = 2, \cdots, n$. Thus $(x, u_1)(y, v_1) - (x, v_1)(y, u_1) = 0$; and, again by the lemma, $\Pi$ contains a non-zero vector orthogonal to $\Pi_1$.

The following corollary is almost immediate:

COROLLARY. *The theorem remains valid if, throughout the statement, we replace $n$ by $m$ and the condition* (1) *by*: (1′) *the $u$'s and $v$'s are the images of a biorthogonal set of vectors under an orthogonal projection of an $m$-dimensional superspace $W$ onto $V$.*

The proof consists of a simple verification of the Parseval relation (*) under these circumstances.

**3. The infinite case.** We now assume that $V$ is real Hilbert space and set $n = \infty$. In this case the theorem remains valid if condition (1) is replaced by:

(1″) *the $u$'s and $v$'s form a biorthogonal set, the $u$'s are complete, and there exist positive constants $m$, $M$ such that*

$$m \sum_{i=1}^{k} \alpha_i^2 \le \sum_{i,j=1}^{k} \alpha_i \alpha_j (u_i, u_j) \le M \sum_{i=1}^{k} \alpha_i^2, \qquad k = 1, 2, \cdots;$$

*or, briefly, the Gramian matrix $((u_i, u_j))$ is bounded above and below.*

The condition of completeness is necessary since, even in the finite case, the theorem is not valid if there are fewer $u$'s than the dimension of $V$. The boundedness condition ensures the Parseval relation (*) and hence a proof of this extended result. Indeed, this follows from Theorems 3, 4, and 8 of [**5**].

Finally, we note that the corollary of Section 2 admits a similar extension to the infinite case.

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## CLASSROOM NOTES

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### ON THE THEOREMS OF CEVA AND MENELAUS

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The following combined proof, based initially on purely projective methods, of the two theorems may be of some interest.

Let $ABC$ (Figure 1) be a triangle lying in a real extended Euclidean plane and $P$ any point in the plane not on a side of the triangle. Let $AP$ meet $BC$ in $X$, $BP$ meet $CA$ in $Y$ and $CP$ meet $AB$ in $Z$. Suppose $DEF$ is a finite straight line in the plane through none of the points $A$, $B$, $C$, or $P$, and let it meet $BC$ in $D$, $CA$ in $E$, $AB$ in $F$. Since not all of the non-collinear points $X$, $Y$, $Z$ can be on $DEF$ we can assume without loss of generality that $X$ does not. Let $DEF$ meet $AP$ in $Q$, $BQ$ meet $AC$ in $Y'$ and $CQ$ meet $AB$ in $Z'$. For the present we will consider all the elements of the configuration to be real.

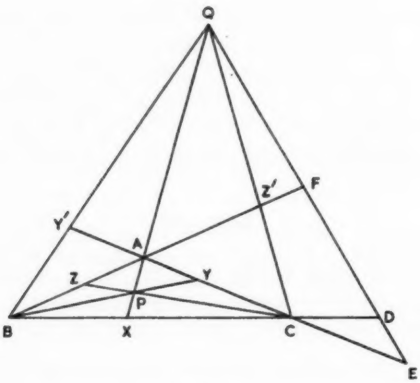



FIG. 1

Projecting the points $Q$, $D$, $E$, $F$ from $A$ onto $BC$, from $B$ onto $CA$ and from $C$ onto $AB$ establishes the equality of the cross-ratios $\{XD, CB\}$, $\{Y'C, EA\}$, $\{Z'B, AF\}$. Further

$$\{XD, CB\} \times \{XC, BD\} \times \{XB, DC\} = \frac{XC \cdot DB}{XB \cdot DC} \times \frac{XB \cdot CD}{XD \cdot CB} \times \frac{XD \cdot BC}{XC \cdot BD} = -1.$$

Since $\{XD, CB\} = \{Y'C, EA\}$ it follows that $\{XC, BD\} = \{Y'E, AC\}$, and since $\{XD, CB\} = \{Z'B, AF\}$ that $\{XB, DC\} = \{Z'F, BA\}$. Hence

$$\{XD, CB\} \times \{Y'E, AC\} \times \{Z'F, BA\} = -1. \tag{1}$$

Projecting the points $Q$, $P$, $A$, $X$ from $B$ onto $CA$ and from $C$ onto $AB$

$$\{Y'Y, AC\} = \{Z'Z, AB\}. \tag{2}$$

By means of a real projection, project $DEF$ into the line at infinity of an

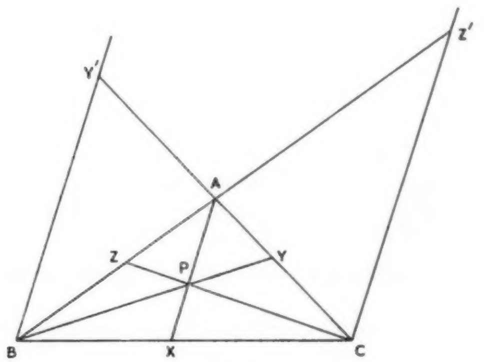



FIG. 2

extended Euclidean plane to obtain Figure 2. Equations (1) and (2) give in Figure 2 the relationships

$$\frac{XC}{XB}\cdot\frac{Y'A}{Y'C}\cdot\frac{Z'B}{Z'A} = -1,$$

$$\frac{Y'A\cdot YC}{Y'C\cdot YA} = \frac{Z'A\cdot ZB}{Z'B\cdot ZA} \quad \text{or} \quad \frac{Y'A}{Y'C}\cdot\frac{Z'B}{Z'A} = \frac{YA}{YC}\cdot\frac{ZB}{ZA}.$$

Hence we deduce Ceva's theorem in Figure 2

$$\frac{XC}{XB}\cdot\frac{YA}{YC}\cdot\frac{ZB}{ZA} = -1,$$

where the point of concurrency, $P$, is any finite point in the plane not on a side of the triangle.

The proof is still valid if, in Figure 1, $DEF$ passes through $Y$ but not through

$Z$ or if through $Y$ and $Z$. The resulting formulas in Figure 2, $XC\cdot ZB/XB\cdot ZA = -1$ and $XC/XB = -1$, merely express a relationship deducible directly from ratios formed by parallels and the bisection by a diagonal of a diagonal of a parallelogram respectively. If, relaxing an initial restriction, $P$ lies on $DEF$ the proof requires only a simple modification. The points $Q$, $Y'$, $Z'$ then coincide with $P$, $Y$, $Z$ respectively in Figure 1. Equation (2) is then non-existent. Equation (1) is still valid and after the projection to Figure 2 leads at once to $XC\cdot YA\cdot ZB/XB\cdot YC\cdot ZA = -1$. $P$ is then a point at infinity in Figure 2 and the formula can also be obtained from the ratios of parallels. In all of these three cases the usual proof of Ceva's theorem implies infinite areas and is not valid.

Returning to Figure 1, the concurrency of the lines $AX$, $BY'$, $CZ'$ in $Q$ leads to the Ceva relation

$$\frac{XC}{YB}\cdot\frac{Y'C}{Y'A}\cdot\frac{Z'A}{Z'B} = -1.$$

The expanded form of equation (1) is

$$\frac{XC\cdot DB}{XB\cdot DC}\times\frac{Y'A\cdot EC}{Y'C\cdot EA}\times\frac{Z'B\cdot FA}{Z'A\cdot FB} = -1.$$

Combining the two expressions we have at once Menelaus' theorem

$$\frac{DB}{DC}\times\frac{EC}{EA}\times\frac{FA}{FB} = 1.$$

(We note here that if Ceva's theorem only is required, the restriction that $DEF$ is a finite line is not needed. Also the restriction that $P$ shall not lie on $DEF$ is only required for procedure to Menelaus' theorem.)

We now discuss the question of unreal elements. Following the development given by Coolidge in his treatise *The Geometry of the Complex Domain*, unreal elements are fundamentally in the projective field. Some care however is needed for the relationships between points lying on an isotropic line. Isotropic lines are so called from their property that their behavior is the same with respect to all rectangular axes through their real point (Greek **ἴσος** equal, *τρόπος* direction or course. Compare the use of the word also in Physics and Biology). In the following discussion we shall slightly vary from Coolidge's notation.

Consider points on the line $y = R\iota x$ where the axes and $R$ are real. We denote the points by the symbols 1, 2, 3, etc. and their $x$ coordinates by $x_r + \iota x_r'$ ($r = 1, 2, 3$ etc.) where $x_r$, $x_r'$ are real. The distance function for points 1, 2 is then ${}_1d_2 = \sqrt{1-R^2}\{(x_2 - x_1)^2 - (x_2' - x_1')^2 + 2\iota(x_2 - x_1)(x_2' - x_1')\}$, or $\sqrt{1-R^2}\{(x_2+\iota x_2') - (x_1+\iota x_1')\}$. Hence if $R \neq \pm 1$, the ratio between the distance functions ${}_1d_2$ and ${}_3d_4$ is independent of $R$. Varying $R$ is effectively projecting the points from one line to other lines with vertex the point at infinity on the $y$-axis. This projection not only gives the usual invariant projective properties among a range of points but also the invariance of the ratio between any two corresponding distance functions. For the isotropic lines through the origin $R = \pm 1$ and the distance function is always zero. If we accept the viewpoint that the invariances persist in these limiting cases, the restriction as to the reality of the elements of the configuration in the real plane can be completely relaxed.

A two-way linkage can also be obtained as follows, but the configuration does not contain an independent projective proof of either theorem. With the triangle $ABC$ and concurrent lines $APX$, $BPY$, $CPZ$ as before, let $YZ$ meet $BC$ in $U$, $ZX$ meet $CA$ in $V$ and $XY$ meet $AB$ in $W$. Since the triangles $ABC$, $XYZ$ are in perspective they are also coaxal and hence $U$, $V$, $W$ are in a straight line. From the quadrangle $AYPZ$ the cross-ratio $\{XU, CB\} = -1$, and similarly for $\{YV, AC\}$, $\{ZW, BA\}$. Multiplying these together and rearranging in expanded form

$$\frac{XC}{XB}\cdot\frac{YA}{YC}\cdot\frac{ZB}{ZA} = -\frac{UC}{UB}\cdot\frac{VA}{VC}\cdot\frac{WB}{WA}$$

and the truth of Menelaus' theorem for the line $UVW$ implies that of Ceva's theorem for any point $P$ not on a side of the triangle. Similarly, using the dual (reciprocal) figure and proof, the truth of Ceva's theorem leads to that of Menelaus for any line not through a vertex of the triangle.

## GRAPHICAL SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS*

J. P. BALLANTINE, University of Washington

**1. Introduction.** Lately,† graphical solutions have appeared for the linear differential equation

$$y' + P(x)\cdot y = Q(x). \tag{1}$$

The following procedure has certain practical and theoretical advantages over those that have appeared.

**2. The method.** First write the equation in the form

$$y' = (B(x) - y)/(A(x) - x), \tag{2}$$

where $A(x) = x + 1/P(x)$, $B(x) = Q(x)/P(x)$.

It is easily seen from equation (2) that $y'$ is the slope of the straight line through the two points $(A(x), B(x))$ and $(x, y)$. This fact makes use of the "parallel ruler" unnecessary.

Set up a coordinate system showing the initial point, $P = (x_0, y_0)$, through which the solution must pass. Rule off a number of strips, parallel to the $y$-axis,

* Presented to a meeting of the mathematicians of the Northwest, Vancouver, B. C., April 2, 1939.

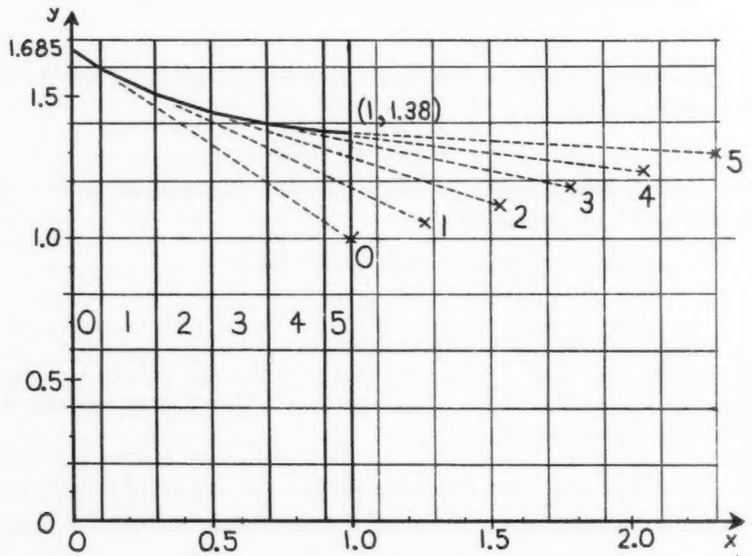
† M. E. Levenson, On a graphical solution, this MONTHLY, vol. 63, 1956, pp. 115–116.

and of width $w$. Let the initial strip, the one numbered 0, have its center along the line $x=x_0$. The center of the $j$th strip will be the line $x=x_j=x_0+jw$. To save ruling lines, use a graph paper on which the scale is so chosen that the necessary rulings are already on the paper.

To each strip will belong a point, $(A, B)$. The point belonging to the $j$th strip is $(A_j, B_j)$, where $A_j=A(x_j)$ and $B_j=B(x_j)$. Let $y=f(x)$ be any solution of differential equation (1). Then the tangent to $y=f(x)$ drawn at the center of any strip passes through the point belonging to that strip. It is helpful to number the strips and the corresponding points.

**3. The solution.** When the strips and corresponding points have been laid off, the solution for any given initial conditions, $x=x_0$, $y=y_0$, is immediate. For simplicity, let $P_0=(x_0, y_0)$ be at the center of strip 0. Start with your ruler passing through $P_0$ and $(A_0, B_0)$. Guided by the ruler, draw the straight line through $P_0$ to the nearer edge of Strip 1. Taking care not to remove the pencil from the paper, turn the ruler so that it still rests against the pencil, but now passes through $(A_1, B_1)$. Now extend the line you have started, completely across Strip 1. Similarly, as you cross Strip 2, the ruler is in line with the point $(A_2, B_2)$ belonging to that strip.

It takes only a few seconds to extend the "solution" across each strip, so that the entire solution is very quickly found, even if the strips are narrow and numerous.



**4. Example.** $y'+(1+0.3x)^{-1}y=1$.

*Solution.* First put the differential equation in the form (2),

$$y' = ((1 + 0.3x) - y)/((1 + 1.3) - x).$$

Thus, $A(x)=1+1.3x$ and $B(x)=1+0.3x$.

The diagram shows Strips 1, 2, 3, 4, each of width 0.2, the last half of Strip 0 and the first half of Strip 5. This will take the solution up to $x=1$.

The values of $A(x)$ and $B(x)$ are then tabulated:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
| $A(x)$ | 1.00 | 1.26 | 1.52 | 1.78 | 2.04 | 2.30 |
| $B(x)$ | 1.00 | 1.06 | 1.12 | 1.18 | 1.24 | 1.30 |

The points $(A_j, B_j)$ are then plotted and numbered.

The diagram shows how the solution, really a broken line, starts at (0, 1.685), with each link in line with the corresponding point $(A_j, B_j)$. It ends at about (1, 1.38).

The theoretical solution is slightly complicated,

$$y = (10/13)(1 + 0.3x) + 0.91577(1 + 0.3x)^{-10/3},$$

and for $x=1$, $y=1.3819$.

I have found this method very useful, not only for solving linear differential equations, but also for plotting such curves as $y=Ke^{rx}$ and $y=Cx^n$ for given values of $K$, $r$, $C$, and $n$. Here $y=Ke^{rx}$ is a solution of $y'=ry=(0-y)/((x-1/r)-x)$, and $y=Cx^n$ is a solution of $y'=ny/x=(0-y)/((1-1/n)x-x)$. In each case, $B(x)\equiv 0$ and $A(x)$ is a simple expression.

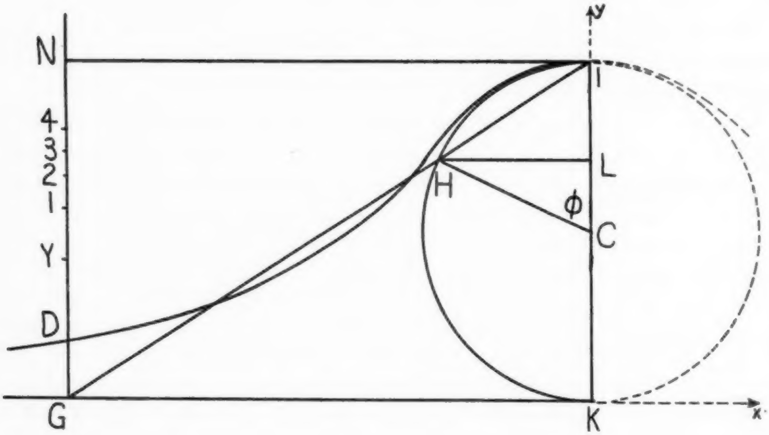
## THE NAMES OF THE CURVE OF AGNESI*

T. F. Mulcrone, S.J., Loyola University, New Orleans

As Vacca first noted [1] it is to the Camaldolese monk, Guido Grandi, that we owe the naming of the locus $x^2y=a^2(a-y)$ which is associated with the name of Maria Gaetana Agnesi (1718–1799). Grandi, in his book ([2], p. 15), gave the name *Scala*, the scale curve, to this locus because it can serve as a measure of light intensity, and in the same work (p. 5, Propositio III) he wrote, "Given a semicircle of diameter $IK$, $\cdots$ the tangent $KG$, $\cdots$ (and) $IG$ intersecting

* Presented to the Louisiana-Mississippi Section of the Mathematical Association of America, February 17, 1956.

the periphery at $H$. This determines the sine $HL$. Let $(GK)^2$ be to $(KI)^2$ as the diameter is to $YN$, and this to $1N$. In this way is had the infinity of terms $2N$, $3N$, $4N$, etc. I affirm that the sum of all the differences of these terms taken




alternately $Y1, 23, 45$, etc. equals the versed sine $IL$ of the intercepted arc $IH$." Taking $\angle ICH=\phi$, $KI=a$, $KG=x$, with $GD=LI=y$ ([2], p. 7, Propositio IV), this means $YN=a^3/x^2$, $1N=a^5/x^4, \cdots$; $Y1=a^3/x^2-a^5/x^4, \cdots$; and $IL=a^3/(x^2+a^2)=a\left[(a/x)^2-(a/x)^4+(a/x)^6-\cdots\right]$

$$= (a/2)(1-\cos\phi) = (a/2)\text{ vers }\phi. \qquad (1)$$

(If $a=2$ we have the unit circle representation of vers $\phi$.)

In 1718, Grandi returns to this curve, now as a "scale of velocities . . . that curve which I describe in my book of quadratures, proposition 4, derived from the versed sine, which I am wont to call the *Versiera* but in Latin (is) *Versoria*." [3]. In (1) Grandi had sufficient justification for this terminology. I find no evidence that he was motivated by the fact that in literary Latin *versoria* denotes "a rope that guides a sail," and *sinus* may mean "the bend or belly of a sail swollen by the wind."

The work of Grandi (and earlier mention by Fermat, and later by Newton) did not attract much interest to the curve. It came to the notice of mathematicians principally through the influential *Instituzioni analitiche* (1748) of Agnesi, the first volume of which was translated into French in 1775. A complete English translation was made by J. Colson of Cambridge in 1801. It is due to this important work of Agnesi that her name became associated with the curve [4].

What is the origin of the name "Witch of Agnesi"? The substantive *versiera* is a synonym for the substantive adjective *versoria*, "turning in every direction," a word derived from the Latin *vertere*, "to turn." In the course of time *versiera* took on another meaning in this way. The Latin words *adversaria*, and by aphaeresis, *versaria*, signify a female who is contrary, an adversary; and in Ecclesiastical Latin one of the added meanings was a female who is contrary to God. Thus, even in the literary Latin of the Middle Ages, the word *versiera* came to be applied, although comparatively rarely, to the one par excellence who is contrary to God, that is, the devil: "a female fiend or goblin," "the devil's grandmother," and other related meanings, the equivalent of the English word witch.

Although Colson was presumably the first to use the word witch in connection with this curve, it should not be supposed that the sinister implication consequent on the use of this word is to be attributed to a mistaken or facetious translation. In his *Analytical Institutions* of Agnesi (1801, vol. 1, p. 222) we read: " . . . the curve to be described, which is vulgarly called the Witch." This translation departs from the original which has: " . . . which is called *Versiera*" [**5**]. Thus it appears that as early as 1801 the technical meaning of *Versiera*, intended by Grandi and Agnesi, had a competitor in Italy among mathematicians in the nontechnical, ecclesiastical connotation of the term.

When we advert to the sinister implication of the word witch, especially when associated with the name of a particular woman, its inappropriateness, even as a mathematical term, is apparent. Far from honoring Agnesi, the ominous term would seem rather to discredit the life and work of a most remarkable and honorable woman—a truly remarkable linguist, mathematical author, correspondent and translator of scholars, serious student of philosophy and theology, and zealous hospital nun.

It is suggested that American and English mathematicians, in their desire to honor the memory of Agnesi, abandon the use of the term "Witch of Agnesi," adopting instead the practice of the French (courbe d'Agnesi) and the Germans (Agnesische Kurve), writing simply "the curve of Agnesi."

The author gratefully acknowledges the generous assistance of the referee and of the Rev. J. F. Moore, S.J.

## References

**1.** G. Vacca, Sulla versiera, Bolletino di bibliografia e storia delle scienze matematiche, Turin, vol. 4, 1901, pp. 33–34.

**2.** G. Grandi, Quadratura Circuli, et Hyperbolae Per Infinitas Hyperbolas, et Parabolas Quadrabiles Geometrice exhibita et demonstrata, 2nd ed. Pisa, 1710, p. 15. (I understand that this material is also in the first edition (1703) of the work.)

**3.** ———, Opere di Galileo Galilei; Note al Trattato del Galileo del moto naturalmente accelerato, Florence, 1718, Tome 3, p. 393, Corollario VI: " . . . nata da'seni versi, che da me suole chiarmarsi la *Versiera* in latino però *Versoria*."

**4.** J. L. Coolidge, Six female mathematicians, Scripta Math., vol. 17, 1951, pp. 20–31.

**5.** M. G. Agnesi, Instituzioni analitiche ad uso del la gioventù italiana, Milan, 1748, Tome 1, p. 280, Problema III. 238: " . . . alla curva da discriversi, che dicesi la Versiera."

**6.** Hugo Beran, The witch of Agnesi, Scripta Math., vol. 8, 1941, p. 135.

## ON DEGENERATE CONICS

J. W. LASLEY, JR., University of North Carolina

**Introduction.** One learns in elementary analytic geometry that the real conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

is elliptic, hyperbolic, parabolic, according to whether the discriminant $D = ab - h^2$ of the quadratic terms is positive, negative, or zero.

One learns further that the conic is composite (degenerate) if, and only if, the discriminant of the conic

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

is, or is not, zero.

**Metric transformations.** It is pointed out that all conics except the proper parabola have at least one center, and that by translation to a center the second degree terms are preserved and the linear terms are deleted. By following this translation by a suitable rotation, one may then remove the product term. Every conic, except the proper parabola, may in this manner be reduced to the simple (canonical) form

$$ax^2 + by^2 + c = 0 \tag{2}$$

of a modified sum of squares.

**Projective transformations.** In projective geometry it is pointed out that by the mere device of choosing a self-polar triangle as the triangle of reference, all conics, including the proper parabola, are capable of being transformed into form (2) in this way.

**An orthogonal transformation.** The transformation employed in the projective case is not usually orthogonal. This paper purports to review the case for the degenerate conics, and to show that an orthogonal transformation can be devised to reduce the degenerate conic to form (2) by a single transformation, which will provide a little used criterion for the determination of the sort of degenerate conic it is.

Let us ask whether we can find a transformation

$$\begin{aligned} x &= \lambda_1 x' + \lambda_2 y' + \lambda_3 z', \\ y &= \mu_1 x' + \mu_2 y' + \mu_3 z', \\ z &= \nu_1 x' + \nu_2 y' + \nu_3 z', \end{aligned} \tag{3}$$

where the conditions

$$
\begin{aligned}
a\lambda + h\mu + g\nu &= k\lambda, \\
h\lambda + b\mu + f\nu &= k\mu, \qquad (4)\\
g\lambda + f\mu + c\nu &= k\nu,
\end{aligned}
$$

hold for each set $(\lambda_i, \mu_i, \nu_i)$, $(i=1, 2, 3)$ and an appropriate $k_i=1, 2, 3$, one for each set.

In order to address ourselves to this question, let us consider the conic in the homogeneous form

$$F = ax^2 + 2hxy + by^2 + 2gxz + 2fyz + cz^2 = 0 \tag{5}$$

and solve the characteristic equation

$$\begin{vmatrix} a-k & h & g \\ h & b-k & f \\ g & f & c-k \end{vmatrix} = 0 \tag{6}$$

for $k=k_1, k_2, k_3$. These values $k$ satisfy (6) and make (4) consistent. The solutions $(\lambda_i, \mu_i, \nu_i)$ from (4), one set for each $k_i$, may or may not make (3) orthogonal. If they do not, they can be made to do so by dividing each set by the square root of the sum of the squares of the three numbers in the set, and taking the numbers so obtained for the coefficients in (3).

**Invariants.** Under transformation (3), now orthogonal, $F$ is an invariant (covariant). So also is $G=x^2+y^2+z^2$. The linear combination $F-kG$ is a further covariant, which because there is no trace of (3) in $k$, makes the coefficients of the characteristic equation

$$k^3 - \sum a\cdot k^2 + \sum A\cdot k - \Delta = 0 \tag{7}$$

invariant also. We thus have

$$\sum a' = \sum a, \qquad \sum A' = \sum A, \qquad \Delta' = \Delta. \tag{8}$$

But the transformation (3) applied to (5) reduces it to

$$a'x'^2 + b'y'^2 + c'z'^2 = 0 \tag{9}$$

where $a'$, $b'$, $c'$ are the solutions $k$ of (7). This applied to (8) gives

$$\sum A = b'c' + a'c' + a'b', \qquad \Delta = a'b'c'. \tag{10}$$

**Degenerate conics.** If our conic is degenerate, $\Delta=0$. If $c'=0$, then $\sum A = a'b'$ and (9) becomes

$$a'x'^2 + b'y'^2 = 0 \tag{11}$$

where $a'$ and $b'$ are solutions of

$$k^2 - \sum a\cdot k + \sum A = 0. \tag{12}$$

It follows at once that if $\sum A$ is positive, $a'$ and $b'$ have the same sign and the lines given by (11) are conjugate complex. If $\sum A$ is negative, the lines given by (11) are real and distinct; if $\sum A$ is zero, the lines are real and coincident.

Now because of the symmetry present in (5), (8), and (9), the case $b'=0$ (or $a'=0$) is not essentially different from the foregoing case in which $c'=0$. For example, if $b'=0$, $\sum A=a'c'$, equation (11) becomes $a'x'^2+c'z'^2=0$. The conclusions stated at the end of the preceding paragraph still hold; only this time in metric cases we have conjugate complex parallel lines, coincident ideal lines, and real and distinct parallel lines as new features—geometric loci hardly obtainable as conic sections, but certainly possible graphs of equation (5), if not of equation (1). Thus, for the conventional conics we take $c'=0$. In this case there is no distinction between $D$ and $\sum A$, since both are $a'b'$. Symmetrically, if $b'=0$, $\sum A$ is the same as $B$; if $a'=0$, $\sum A$ is the same as $A$. In all cases, $\sum A$ plays the role of the discriminant of a quadratic form in two variables. In all cases, the conclusions reached above obtain: If $\sum A$ is positive, the lines are conjugate complex; if negative, the lines are real and distinct; if zero, the lines are coincident. This with the understanding that equations of the form $px+qy+rz=0$ represent a straight line, provided $p$, $q$, $r$ are constants and $(p, q, r)\neq(0, 0, 0)$.

**Summary.** Thus we see that for the determination of the type of degenerate conics we have three criteria. The discriminant of the quadratic form $D=ab-h^2$ for telling the conics from which the degenerate conics degenerate. If $D$ is positive, we have elliptic lines; for example, $x^2+y^2=0$. If $D$ is negative, we have hyperbolic lines; for example, $xy=0$. If $D$ is zero we have parabolic lines; for example, $x^2=0$, $x^2-1=0$, $x^2+1=0$.

The rank $r$ of the conic tells us whether the lines are distinct (rank 2) or coincident (rank 1); for example, $xy=0$, $x^2+y^2=0$, $x^2-1=0$, $x^2+1=0$ are distinct lines. No considerations of reality are made here. If $r=1$, the lines are coincident; for example, $x^2=0$.

In the case of the criterion $\sum A$, the trace of the adjoint matrix of the conic: if $\sum A$ is positive, we have a pair of complex lines; for example, $x^2+y^2=0$, $x^2+1=0$. If $\sum A$ is negative we have a pair of real and distinct lines; for example, $xy=0$, $x^2-1=0$. If $\sum A$ is zero, we have a repeated real line; for example, $x^2=0$.

The trace criterion thus distinguishes conjugate complex lines, for which $\sum A$ is positive, from real and distinct lines, for which $\sum A$ is negative. It distinguishes distinct lines, for which $\sum A\neq 0$, from repeated lines, for which $\sum A=0$. Moreover, it distinguishes conic sections—those which may be actually cut from the cone; for example, $xy=0$, $x^2+y^2=0$, $x^2=0$, from the graphs of the equation of the second degree—to which must be added degenerate conics such as $x^2-1=0$, $x^2+1=0$, not obtainable by cutting a cone.

## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTION

E 1266. *Proposed by D. C. B. Marsh, Colorado School of Mines*

Solve

$$a^3 - b^3 - c^3 = 3abc, \qquad a^2 = 2(b + c)$$

simultaneously in positive integers.

E 1267. *Proposed by Ivan Niven, University of Oregon*

The divergence of the harmonic series $\sum 1/n$ is often established by comparison with the obviously divergent series $\sum f(n)$ where $f(n) = 2^{-k}$, the integer $k$ being defined by the inequality $2^k \geqq n > 2^{k-1}$. Establish the convergence or divergence of the series $\sum(1/n - f(n))$.

E 1268. *Proposed by A. J. Goldman and P. S. Wolfe, Princeton University*

Evaluate the determinant $D_n$ which has $(1, 2, \cdots, n)$ as first row, $(2, 3, \cdots, n, 1)$ as second row, *etc.*

E 1269. *Proposed by Frank Kocher, Pennsylvania State University*

Prove that the area under one arch of the curve generated by a vertex of a regular polygon rolling on a straight line is equal to the area of the polygon plus twice the area of its circumscribed circle.

E 1270. *Proposed by Leo Moser, University of Alberta*

What is the smallest positive even integer $n$ such that in both $n$ and $n+1$ dimensions the regular simplex of edge 1 will have a rational number as its content. (Dedicated to Professor H. S. M. Coxeter.)

### SOLUTIONS

#### A Linear Diophantine Equation

E 1236 [1956, 664]. *Proposed by Hazel E. Evans, University of Pittsburgh*

For $a > b$ and $N < ab$ find the maximum value of $N$ for which the equation

$$ax + by = N$$

has a solution in non-negative integers.

*Solution by E. D. Schell, Remington Rand Univac, New York.* We assume that it is intended that $a$ and $b$ represent positive integers. Now suppose $K=\max N$ subject to the stated conditions. Then $ax+by=K$, and $(a, b)$ divides $K$. But the largest $N<ab$ for which this could be true is $ab-(a, b)$.

Set $-(a, b)=\alpha a-\beta b$, where $\alpha<b$, $\beta<a$, and $\alpha, \beta>0$, by using the Euclidean algorithm. Add $ab$ to each side, obtaining $ab-(a, b)=\alpha a+(a-\beta)b$. Then $x=\alpha$ and $y=a-\beta$ are non-negative solutions for $K$.

Also solved by D. A. Breault, J. C. W. De la Bere, Underwood Dudley, A. R. Hyde, Sidney Kravitz, D. C. B. Marsh, J. B. Muskat, E. N. Nilson, W. L. Ostrowski, Azriel Rosenfeld, D. J. Schaefer, and the proposer. Late solutions by J. W. Harter, J. H. Hodges, and R. H. Hou.

*Editorial Note.* If $a$ and $b$ are taken to represent *any* integers, then the following facts can be established: (1) if $a>b=0$, there is no solution; (2) if $a>0>b$, then $K=ab-(a, b)$; (3) if $0=a>b$, then $K=b$; (4) if $0>a>b$, then $K=0$.

### A Pair of Line Integrals

E 1237 [1956, 664]. *Proposed by Viktors Linis, University of Ottawa*

Let $E$ be an ellipse, $r_1$ and $r_2$ focal radii, $\alpha$ the angle between the focal radii, and $ds$ the element of arc. Evaluate the integrals

$$\int_E ds/(r_1r_2)^{1/2} \quad\text{and}\quad \int_E (\cos \alpha/2)ds.$$

*Solution by Chih-yi Wang, University of Minnesota.* Let the parametric representation of $E$ be $x=a\cos\theta$, $y=b\sin\theta$, $a>b>0$, $0\leqq\theta<2\pi$. Then we have

$$ds = [a^2 - (a^2 - b^2)\cos^2\theta]^{1/2}d\theta,$$

$$r_1 = a - (a^2 - b^2)^{1/2}\cos\theta,$$

$$r_2 = a + (a^2 - b^2)^{1/2}\cos\theta,$$

whence

$$\int_E ds/(r_1r_2)^{1/2} = \int_0^{2\pi} d\theta = 2\pi.$$

For the second integral, we make use of the cosine law and the half angle formula to obtain

$$\cos\alpha/2 = b/[a^2 - (a^2 - b^2)\cos^2\theta]^{1/2},$$

whence

$$\int_E (\cos\alpha/2)ds = \int_0^{2\pi} b\,d\theta = 2\pi b.$$

Also solved by J. C. W. De la Bere, David Freedman, A. R. Hyde, J. B. Johnston, M. S. Klamkin, C. S. Ogilvy, C. D. Olds, L. A. Ringenberg, Jeff Ritterman, Azriel Rosenfeld, Nathan Shklov, A. V. Sylwester, David Zeitlin, and the proposer.

**Three Consecutive Powers of 3**

E 1238 [1956, 665]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Determine integral values of $n>0$ such that $3^n$, $3^{n+1}$, $3^{n+2}$ all have the same number of digits in their denary expansions.

*Solution by Joe Lipman, University of Toronto.* If $n$ is an integer such that $3^n$, $3^{n+1}$, $3^{n+2}$ all have the same number of digits in their denary expansions, then

$$\underbrace{10000\cdots}_{k \text{ digits}} < 3^n < \underbrace{11111\cdots}_{k \text{ digits}}.$$

Now the mantissa of $\log 11111\cdots$ is $0.0457574\cdots$. If the inequality is satisfied, $n \log 3 =$ an integer + a decimal fraction between zero and 0.0457574. But $\log 3 = 0.477121256$, which is just slightly greater than 1/21. Therefore we can expect the $n$'s to recur at intervals of 21 or 23. Thus we have

| $n$ | $n \log 3$ |
|---|---|
| 21 | 10.019546376 |
| 42 | 20.039092152 |
| 65 | 30.01288163 |
| 86 | 41.03242802 |
| 109 | 52.00621 $\cdots$ |
| 130 | 62.02575 $\cdots$ |
| 151 | 72.04529 $\cdots$ |
| 174 | 83.0190985 |

A comparison of 174 and 21 shows that the corresponding mantissae differ by only 0.0004478. This is because $153 \log 3 = 72.9995522$. Thus any number of the form $21+153k$, where $k<19546376/4478=43.8\cdots$ will be one of the required $n$'s. For $k=44$, the resulting mantissa is 0.9998432. Subtracting $0.9542425 = 2 \log 3$, we get the mantissa 0.0456007. So instead of $21+153(44)$, use $21+153(44)-2 = 6751$. Now using the sequence $6751+153k_1$, where $k_1<456007/4478$, repeat the above process to get more $n$'s. Then derive the new sequence $22355+153k_2$, and so on. In this way arbitrarily large $n$'s can be determined as long as tables of sufficient accuracy are available.

Also solved by J. C. W. De la Bere, Monte Dernham, Hazel E. Evans, Michael Goldberg, A. R. Hyde, I. M. Isaacs, Sidney Kravitz, D. C. B. Marsh, Herbert Nadler, C. S. Ogilvy, D. S. Passman, L. A. Ringenberg, Azriel Rosenfeld, E. D. Schell, G. W. Walker, and the proposer.

**Two Related Quadrangles**

E 1239 [1956, 665]. *Proposed by Josef Langr, Prague, Czechoslovakia*

Let $Q' \equiv A'B'C'D'$ be the quadrangle formed by the orthocenters $A'$, $B'$, $C'$, $D'$ of triangles $BCD$, $CDA$, $DAB$, $ABC$ of a given convex quadrangle $Q \equiv ABCD$. Show that: (1) the vertices of $Q$ and $Q'$ lie on a common equilateral hyperbola, (2) $Q$ and $Q'$ have equal areas.

*Solution by D. C. B. Marsh, Colorado School of Mines.* The vertices of $Q$ determine an equilateral hyperbola, which may be taken as $xy=1$ by superimposing a properly scaled coordinate system upon it. We label coordinates as $A$: $(a, 1/a)$, $B$: $(b, 1/b)$, $C$: $(c, 1/c)$, $D$: $(d, 1/d)$. It is a simple matter to find the orthocenters $A'$: $(-1/bcd, -bcd)$, $B'$: $(-1/acd, -acd)$, $C'$: $(-1/abd, -abd)$, $D'$: $(-1/abc, -abc)$, which are obviously co-hyperbolic with $A$, $B$, $C$, $D$, and (1) is established.

Assuming $A$, $B$, $C$, $D$ are the vertices of $Q$ in order, the area of $Q$ is given by the sum of the absolute values of

$$(1/2)\begin{vmatrix} a & 1/a & 1 \\ b & 1/b & 1 \\ c & 1/c & 1 \end{vmatrix} \quad \text{and} \quad (1/2)\begin{vmatrix} c & 1/c & 1 \\ d & 1/d & 1 \\ a & 1/a & 1 \end{vmatrix}.$$

Multiplying the first columns of both determinants by $-1/abcd$ and the second columns by $-abcd$ does not change the numerical value, but the form becomes that of the area of $Q'$, demonstrating (2).

Also solved by K. W. Crain, J. C. W. De la Bere, C. S. Ogilvy, O. J. Ramler, Sister M. Stephanie, and the proposer.

*Editorial Note.* The vertices of a convex quadrangle determine a *nondegenerate* equilateral hyperbola unless the line through one pair of vertices is perpendicular to the line through the other pair. This exceptional case is easily treated either on its own merits or as a limiting situation of the general case treated above.

**Two Six-piece Dissections**

E 1240 [1956, 665]. *Proposed by H. Lindgren, Patent Office, Canberra, Australia*

Find six-piece dissections of a regular dodecagon into a square and into a Greek cross.

*Solution by the proposer.* It is readily verified that a chord subtending four sides of a regular dodecagon is equal to a side of the equivalent square. There are numerous six-piece dissections based on this fact. Those shown in Figures

1 and 2 are perhaps the neatest.

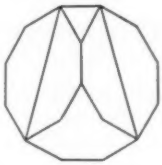


FIG. 1

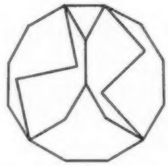
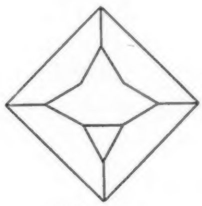
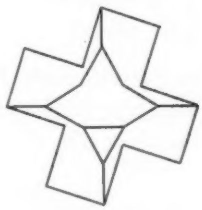


FIG. 2





These dissections were found by a general method described in *The Australian Mathematics Teacher*, vol. 7, 1951, pp. 7–10, vol. 9, 1953, pp. 17–21, 64.

## ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.*

### PROBLEMS FOR SOLUTION

4738. *Proposed by R. R. Goldberg, Pittsburgh, Pa.*

If, for all positive $x$, $\sum_{k=1}^{\infty} |F(kx)| < \infty$ and $\sum_{k=1}^{\infty} F(kx)=0$, then $F(x)$ vanishes identically.

4739. *Proposed by V. L. Klee, Jr., University of Washington*

Suppose $C$ is a closed convex subset of the Euclidean space $E^3$ whose boundary is a regular octahedron, and that $C_1$, $C_2$, and $C_3$ are translates of $C$ (i.e.,

$C_i = C + x_i$ for some $x_i \in E^3$). Then, if each of the intersections $C_1 \cap C_2$, $C_2 \cap C_3$, and $C_3 \cap C_1$ is non-empty, must $C_1 \cap C_2 \cap C_3$ be non-empty?

4740. *Proposed by R. J. Dickson, Lockheed Aircraft Corporation, Burbank, California*

Is every locally schlicht analytic mapping of the complex plane onto itself a schlicht mapping?

4741. *Proposed by L. A. Rubel, Institute for Advanced Study*

Prove or disprove the statement: If a metric space $S$ is homeomorphic to its completion, then $S$ is complete.

4742. *Proposed by Joshua Barlaz, Rutgers University*

Evaluate the Cesàro first order mean for the series $\sum_{n=2}^{\infty} (-1)^n \log n$.

## SOLUTIONS

### Functions Restrained by an Integral Inequality

4660 [1955, 659]. *Proposed by E. M. Wright, University of Aberdeen, Scotland*

For all $x \geqq 1$, $f(x)$ and $\phi(x)$ are non-negative functions, bounded and integrable in any finite interval. They satisfy the inequality

$$xf(x) \leqq \int_1^x f(t)dt + \phi(x).$$

(i) If $\int^{\infty} \phi(t)t^{-2}dt = \infty$, find an $f(x)$ such that $f(x) \to \infty$ as $x \to \infty$.
(ii) If $\phi(x)/x \to 0$ and $\int^{\infty} \phi(t)t^{-2}dt < \infty$, show that

$$g(x) = \frac{1}{x}\int_1^x f(t)dt$$

tends to a limit as $x \to \infty$ and that $\lim g(x) = \limsup f(x)$.
(iii) Show that, whatever restriction we may impose on the order of $\phi(x)$ as $x \to \infty$, we cannot thereby ensure that $f(x)$ tends to a limit.

*Solution by R. O. Davies, University College, Leicester, England.* Let $(E)$ denote the inequality.
(i) If $f(x) = \phi(x)x^{-1} + \int_1^x \phi(t)t^{-2}dt$, then $f(x) \to \infty$, and integration by parts shows that $(E)$ holds with equality.
(ii) Integration by parts shows that

$$g(x) = \int_1^x \left[ x^{-1}f(x) - x^{-2}\int_1^x f(t)dt \right] dx.$$

Hence, for $x_1 < x_2$, using $(E)$ multiplied through by $x^{-2}$, we have

$$(1) \qquad g(x_2) - g(x_1) = \int_{x_1}^{x_2} \left[ x^{-1} f(x) - x^{-2} \int_1^x f(t)dt \right] dx \leqq \int_{x_1}^{x_2} \phi(t)t^{-2}dt.$$

Consequently, $g(x)$ is bounded above (as well as below, by zero). It now follows that $g(x)$ tends to a limit. For otherwise by choosing a large value of $x_1$ for which $g(x_1)$ was near its lower limit and a larger value of $x_2$ for which $g(x_2)$ was near its upper limit we could obtain from (1) a contradiction to the convergence of $\int \phi(t)t^{-2}dt$.

That $\lim g(x) \leqq \lim\sup f(x)$ is a standard result, and the reverse inequality follows from $(E)$, since $\phi(x)x^{-1} \to 0$.

(iii) Let $f(x) = x^{-1} + 1$. For all large $x$ we have

$$xf(x) = 1 + x < \log x - 1 + x = \int_1^x f(t)dt,$$

and so $(E)$ will be satisfied with a $\phi(x)$ which is zero for all large $x$. Without violating $(E)$ we may now destroy the convergence of $f(x)$ by changing its value to zero for (say) all large integer values of $x$; or, if we wish, throughout small intervals surrounding them, since there is strict inequality in $(E)$.

Also solved by R. P. Boas, Jr., and the proposer.

**Zeros in a Triple Diagonal Matrix**

4681 [1956, 191]. *Proposed by Jack Klugerman, Evans Signal Laboratory, Belmar, N. J.*

Given a real symmetric matrix $A$ which is triple diagonal, *i.e.*, it has a diagonal, an upper diagonal, a lower diagonal, and the remaining elements are zero; if $c$ is the eigenvalue with highest multiplicity $m$, then there must be at least $m-1$ zeros in the upper diagonal.

*Solution by N. J. Fine, University of Pennsylvania.* Since $A$ is real symmetric, its eigenvalues span $R^n$, so the subspace $V$ corresponding to $c$ has dimension $m$. We can find a basis $v_1, \cdots, v_m$ for $V$ such that $(v_i, e_k) = 0$ for $k \leqq k_i$, $(v_i, e_{k_i+1}) \neq 0$, where the $e_k$ are unit vectors and $1 \leqq k_2 < k_3 < \cdots < k_m$. We then have, for $i = 2, \cdots, m$,

$$\begin{aligned} 0 = c(v_i, e_{k_i}) &= (Av_i, e_{k_i}) = (v_i, Ae_{k_i}) \\ &= (v_i, a_{k_i,k_i-1}e_{k_i-1} + a_{k_i,k_i}e_{k_i} + a_{k_i,k_i+1}e_{k_i+1}) \\ &= a_{k_i,k_i+1}(v_i, e_{k_i+1}). \end{aligned}$$

Thus the $m-1$ upper diagonal elements $a_{k_i,k_i+1}$ are zero. (For a discussion of the concepts used here see, *e.g.*, MacDuffee, *Vectors and Matrices.*)

Also solved by Harley Flanders, Wallace Givens, W. V. Parker, and the proposer.

**Non-rectifiable Simple Closed Curve**

4687 [1956, 259]. *Proposed by John Wermer, Brown University*

Let $\Gamma$ be a simple closed curve in the complex plane containing the origin in its interior. Show that if $\Gamma$ is not rectifiable, then we can approximate the constant 1 uniformly on $\Gamma$ by functions $\sum_{n=-N}^{N} c_n z^n$, $c_0=0$.

*Solution by the proposer.* Let $\phi$ map $|\lambda|<1$ conformally on the interior of $\Gamma$ with $\phi(0)=0$. Suppose we cannot approximate the constant 1 in the indicated fashion. Then we cannot approximate 1 uniformly on the unit circle by functions $\sum_{-N}^{N} c_n\phi^n(\lambda)$, $c_0=0$. By a classical theorem of F. Riesz, there then exists a measure $\mu$ on the unit circle with $\int_{|\lambda|=1}\phi^n(\lambda)d\mu(\lambda)=0$, $n\neq 0$, $\int_{|\lambda|=1}d\mu(\lambda)=1$. The measure $\phi(\lambda)d\mu(\lambda)$ is then orthogonal to all functions $P(\phi(\lambda))$ where $P$ is a polynomial. Since $\phi$ is schlicht, functions $P(\phi(\lambda))$ approximate uniformly on $|\lambda|=1$ to each function $f$ which is continuous in $|\lambda|\leqq 1$ and analytic in $|\lambda|<1$. Hence $\phi(\lambda)d\mu(\lambda)$ is orthogonal to all such functions. By another theorem of F. (and M.) Riesz, this implies that $\phi(\lambda)d\mu(\lambda)=h(\lambda)d\lambda$, where $h(\lambda)$ is the boundary value of a function $h(z)$ analytic in $|z|<1$ and with $\int_0^{2\pi}|h(re^{i\theta})|d\theta$ bounded as $r\to 1$.

On the other hand, if $\phi'$ is the derivative of $\phi$, then

$$\frac{1}{2\pi i}\int_{|\lambda|=r}\phi^n(\lambda)\phi'(\lambda)d\lambda = \frac{1}{2\pi i}\int_{|\lambda|=r}\frac{d}{d\lambda}\left\{\frac{\phi^{n+1}(\lambda)}{n+1}\right\}d\lambda = 0, \qquad n\neq -1$$

and

$$\frac{1}{2\pi i}\int_{|\lambda|=r}\phi^{-1}(\lambda)\phi'(\lambda)d\lambda = 1$$

for each $r<1$. Now also

$$\int_{|\lambda|=r}\phi^n(\lambda)h(\lambda)d\lambda = 0, \quad n\neq -1, \qquad \int_{|\lambda|=r}\phi^{-1}(\lambda)h(\lambda)d\lambda = 1.$$

Hence for all $n$

$$\int_{|\lambda|=r}\phi^n(\lambda)\left\{h(\lambda)-\frac{1}{2\pi i}\phi'(\lambda)\right\}d\lambda = 0.$$

But the functions $\{\phi^n(\lambda)\}_{-\infty}^{\infty}$ are uniformly dense on $|\lambda|=r$ by a theorem of Walsh. Hence $h(\lambda)=\phi'(\lambda)/2\pi i$ on $|\lambda|=r$. This is true for each $r<1$. Hence

$$\frac{1}{2\pi}\int_0^{2\pi}|\phi'(re^{i\theta})|\,rd\theta = \int_0^{2\pi}|h(re^{i\theta})|\,rd\theta$$

is bounded as $r\to 1$. Let $\Gamma_r$ denote the image under $\phi$ of $|\lambda|=r$. Then the lengths of the $\Gamma_r$ are uniformly bounded as $r\to 1$ by the preceding and, also, $\Gamma_r$ converges to $\Gamma$ as $r\to 1$. Hence $\Gamma$ is of finite length. Hence, if $\Gamma$ is not rectifiable, the approximation must be possible.

**Sums of Distinct Divisors**

4688 [1956, 346]. *Proposed by A. H. Clifford, Tulane University*

What positive integers $n$ have the property that every positive integer less than $n$ is expressible as the sum of distinct divisors of $n$?

I. *Solution by Virginia S. Hanly, Ohio State University.* Let the prime factorization of $n$ be $n=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$, $p_1<p_2<\cdots<p_r$. In order that every positive integer less than $n$ be expressible as a sum of distinct positive divisors of $n$, it is necessary and sufficient that $p_1=2$, $p_{i+1}-1=\sigma(p_1^{a_1}p_2^{a_2}\cdots p_i^{a_i})$ for $i=1, 2, \cdots, r-1$, where $\sigma(k)$ denotes the sum of all positive divisors of $k$. The necessity is obvious.The proof of sufficiency is by induction on $r$, observing first that if $b$ and $c$ are positive integers, $c\geqq b-1$, then the variable $x_0+x_1b+\cdots+x_kb^k$, $0\leqq x_i\leqq c$, assumes each of the values $0, 1, \cdots, c(b^{k+1}-1)/(b-1)$. Now let the proposed set of conditions be valid for $q_r=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$. We assert that every positive integer not greater than $\sigma(q_r)$ is expressible as a sum of distinct divisors of $q_r$. The assertion is clearly true for $r=1$. Any number $x_0+x_1p_r+\cdots+x_{a_r}p_r^{a_r}$, where each $x_i$ is a sum of distinct divisors of $q_{r-1}$, is a sum of distinct divisors of $q_r$. Thus, if our assertion is true for $r-1$ it is true for $r$.

II. *Note by Bernard Jacobson, Michigan State University.* If positive divisors only are to be used, then the complete solution is given by B. M. Stewart (*Sums of distinct divisors*, Amer. J. Math. vol. 76, 1954, pp. 779–785, Corollary 1). If it is permitted to use also negative divisors, then a similar analysis will show that the numbers $n$ have the prime factorization

$$n = 2^b3^c\prod_{i=1}^{k} p_i^{a_i}, \qquad 3 < p_1 < \cdots < p_k,$$

subject to the conditions $p_1-1\leqq 2\sigma(2^b3^c)$, $p_j\leqq 2\sigma(2^b3^c\prod_{i=1}^{j-1}p_i^{a_i})$ for $j=2, \cdots, k$, and $b$ and $c$ are not both zero. (This result was communicated to the American Mathematical Society, 1956. See *Abstract* 407, Bull. Amer. Math. Soc. vol. 62, 1956, p. 351.)

Also solved by A. S. Davis, J. P. Mayberry, D. C. B. Morse, P. P. Saworotnow, and the proposer.

**Limit of a Class of Sums**

4689 [1956, 346]. *Proposed by D. J. Newman, AVCO Research Division, Lawrence, Mass.*

Let $f(x)$ be any function such that $f'''(x)\geqq 0$, $f(n)\sim f(n+1)$. Prove that $\sum_{n=0}^{\infty}(-1)^n t^{f(n)}$ tends to $\frac{1}{2}$ as $t\to 1^-$.

*Indications by the proposer.* Assuming $f(0)=0$ and $f'(0), f''(0)\geqq 0$, the problem is equivalent to showing that

$$\lim_{t\to 1^-}\{t^{f(0)} - 2t^{f(1)} + 2t^{f(2)} - \cdots\} = 0.$$

The last expression (the series being absolutely convergent for $|t| < 1$) may be written in the form

$$(t^{f(0)} - 2t^{f(1)} + t^{f(2)}) + (t^{f(2)} - 2t^{f(3)} + t^{f(4)}) + \cdots = \sum_{n=0}^{\infty} \Delta^2 t^{f(n)},$$

where $\Delta^2 F(n) = F(2n) - 2F(2n+1) + F(2n+2)$. From the mean value theorem we have $\Delta^2 F(n) = F(2n+\theta_n)$, $0 \leqq \theta_n \leqq 2$. Therefore

$$\sum_{n=0}^{\infty} \Delta^2 t^{f(n)} \leqq \log^2 \frac{1}{t} \sum [f'(2n+2)]^2 t^{f(2n)} + \log \frac{1}{t} \sum t^{f(2n)} f''(2n+2),$$

and the last two sums approach zero as $t \to 1^-$.

### Two Tetrahedrons and an Invariant

4690 [1956, 346]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Having given a tetrahedron $ABCD$ and the tetrahedron $A_1B_1C_1D_1$ obtained by passing planes through $A$, $B$, $C$, $D$ parallel to the opposite faces of $ABCD$, show that

$$PA^2 + PB^2 + PC^2 - 2PD^2 - PD_1^2$$

is a constant independent of the position of point $P$. Extend this property to a skew polygon of $n$ vertices.

*Solution by N. A. Court, University of Oklahoma.* Considering the general case first, let $A_i$ $(i = 1, \cdots, n)$ be $n$ given points in space having $G$ for their centroid, and let $k$ denote the sum of the squares of the $n(n-1)/2$ segments determined by the $n$ given points. If $P$ is any point in space, we have:

$$\sum_{i=1}^{n} PA_i^2 = \sum_{i=1}^{n} GA_i^2 + nPG^2 \qquad [1;\text{ p. 316, art. 274}], \tag{1}$$

$$k = n \sum_{i=1}^{n} GA_i^2 \qquad [1;\text{ p. 321, art. 280}], \tag{2}$$

whence

$$\sum_{i=1}^{n} PA_i^2 = nPG^2 + k/n. \tag{3}$$

If $E$ is a point on the line $GA_n$, and $EG:GA_n = t$, both in magnitude and in sign, we have, by Stewart's theorem [2]:

$$PE^2 \cdot GA_n + PG^2 \cdot A_nE + PA_n^2 \cdot EG + GA_n \cdot A_nE \cdot EG = 0. \tag{4}$$

Now $EG = tGA_n$, and $A_nE = -(t+1)GA_n$, hence (4) becomes, after division by $GA_n$,

$$PE^2 - (t+1)PG^2 + tPA_n^2 - t(t+1)GA_n^2 = 0. \tag{5}$$

Eliminating $PG^2$ between (3) and (5), the result may be put in the form

$$\sum_{i=1}^{n} PA_i^2 - n(PE^2 + tPA_n^2)/(t+1) = k/n - ntGA_n^2. \tag{6}$$

The right hand side of (6) is a constant, independent of the position of $P$, and this constant is the value of the left hand side, which proves the proposition in the general case. Observe that this proposition is valid in Euclidean space of any number of dimensions.

In the case of the tetrahedron $(T) = ABCD$, the vertex $D_1$ of the anticomplementary tetrahedron $(T_1) = A_1B_1C_1D_1$ of $(T)$ lies on the line $GD$, where $G$ is the centroid of $(T)$, and $D_1G = 3GD$ [3: p. 53, art. 176]. Thus the vertex $D_1$ may in the present case play the role of the point $E$ of the general case, and we have: $=3$, $n=4$, and (6) becomes

$$PA^2 + PB^2 + PC^2 + PD^2 - 4(PD_1^2 + 3PD^2)/4 = k/4 - 12GD^2,$$

or

$$PA^2 + PB^2 + PC^2 - 2PD^2 - PD_1^2 = k/4 - 12GD^2,$$

where $k$ is the sum of the squares on the six edges of $(T)$.

### References

1. L. N. M. Carnot, *Geometrie de Position*, Paris, 1803.
2. Nathan Altshiller-Court, *College Geometry*, 2nd ed., New York, 1952.
3. ———, *Modern Pure Solid Geometry*, New York, 1935.

Also solved by G. B. Robison, and the proposer.

### Minimal Weakly Prime Ideal

4691 [1956, 347]. *Proposed by R. E. Johnson, Smith College*

A weakly prime ideal of a ring $R$ is any ideal $I$ having the property that either $aR \subset I$ or $Ra \subset I$ implies that $a$ is in $I$. Give an element-wise characterization of the unique minimal weakly prime ideal of $R$.

*Solution by Alfredo Jones, Instituto de Matematica y Estadistica, Montevideo, Uruguay.* Given any ideal $I$, let: $W(I) = \{a: R^naR^m \subset I \text{ for some } m, n > 0\}$. $W(I)$

is obviously an ideal. $W(I)$ is weakly prime because if $AR \subset W(I)$, $R^naRR^m = R^naR^{m+1} \subset I$, so $a \in W(I)$, and similarly if $Ra \subset W(I)$. And if $I$ is weakly prime $W(I) = I$, so we thus obtain all weakly prime ideals. But if $I_1 \subset I_2$ then $W(I_1) \subset W(I_2)$. Therefore the minimal weakly prime ideal is: $W(0) = \{a: R^naR^m = 0$ for some $m, n > 0\}$.

Also solved by D. S. Kahn, and the proposer.

**Topological Space with Unique Limits**

4694 [1956, 426]. *Proposed by R. W. Bagley, University of Kentucky*

There are simple examples which show that an uncountable topological space in which limits are unique (hence $T_1$) is not necessarily Hausdorff. Are there such examples for countable spaces? Here "limit" is used in the usual sense of limit of a sequence where the directed set is the positive integers rather than limit of a generalized sequence as defined by Kelley and others. With this general definition (where the directed set is allowed to vary), Kelley proved that a space is Hausdorff if, and only if, limits are unique. (See *Convergence in topology*, Duke Math. J., 1950, pp. 277–283).

I. *Solution by H. E. Vaughan, University of Illinois.* Frechet, in his book *Les Espaces Abstraits*, pp. 212–213, attributes the following example to Urysohn. Let $R$ consist of the rational numbers belonging to the closed interval $[0, 1]$ together with an irrational number, and let convergence be defined as follows: A sequence of points of $R$, which, with respect to the usual topology of the real line, converges to a rational number is to converge to the same limit in $R$; a sequence which ordinarily converges to an irrational number is to converge, in $R$, to the irrational member of $R$. It is readily seen that, with this definition of convergence, $R$ is an $L$-space, and an investigation of the neighborhoods of its points shows that it is also a topological space (in the modern, very restricted, sense of the phrase) which is not a Hausdorff space.

II. *Solution by M. K. Fort, Jr., University of Georgia.* Let $R$ be the set of all rational numbers. We define $T$ to be the set of all subsets $X$ of $R$ such that either $X$ is empty or $R-X$ has at most a finite number of limit points in the real number system relative to the usual topology for the real number system. It is easy to verify that $T$ is a $T_1$ topology for $R$. However, $T$ is not Hausdorff since any two non empty members of $T$ have a non empty intersection. Limits of sequences are unique, since a sequence $x_1, x_2, x_3, \cdots$ converges to a point $p$ relative to this topology if and only if $x_n = p$ for all sufficiently large values of $n$.

Also solved by G. E. Bredon, Helen F. Cullen, L. R. Ford, Jr., Melvin Henriksen, and the proposer.

## RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

*All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.*

*Mathematics of Business, Accounting, and Finance.* By K. L. Trefftzs and E. J. Hills. Harper, New York, 1956. 591 pp. $4.50.

This rather long book is designed to meet the needs of the average and subaverage student beginning a commercial education and "may provide the only college training in mathematics that many students receive." The first 116 pages is a review of the arithmetic usually presented in the first eight grades. The next 134 pages reviews most of the first year's work in high school algebra. The remainder is devoted to elementary business problems, mathematics of finance, and insurance.

The aim of the first two parts seems to be to provide a sufficient amount of drill necessary to make a student proficient in the fundamental arithmetic operations and in basic algebraic techniques. Very little new material is included, the authors preferring the time honored "high school" method for obtaining square roots to Hero's method. No attempt is made to develop the real number system. In fact an illustration of the use of a rule for multiplying rounded numbers (p. 73) implies that the real numbers are not dense and at the same time disproves the rule. In the algebra part only a half page is devoted to fractional and negative exponents, and the binomial theorem and progressions are omitted entirely, as are functions and graphs. There are, however, a large number of problems, but the student is not trusted with his own analysis. Problems in algebra are classified, each type carefully analyzed, and the steps necessary for their solution enumerated.

The student is given little opportunity for analysis in the remainder of the book. The analysis of problems of various types, the statement of the rule to be followed, the enumeration of the steps necessary to produce a solution is expertly done by the authors instead. The student is not asked to use time diagrams, although they occasionally appear to assist the authors in the development of a rule or formula. Equations are written (p. 293) which imply the equivalence of dated payments at a simple interest rate. Since this is not a true equivalence relation, one wonders if students solving problems by this technique may not get different answers if different focal dates are selected. The mere statement (p. 353) of transitivity for equivalence at a compound interest rate seems hardly sufficient for students to master this important concept. "Finding the unknown time" (p. 339) by interpolation in the compound interest table is treated at a time when $(1+i)^n$ is defined only for integral values of $n$, and the resulting answer is called an approximation of the time when in actuality it is the exact time necessary for $P$ to accumulate to $S$ by a later rule (p. 345). Since no knowledge of progressions is assumed the students must accept the formula for the amount of an annuity on faith (p. 395). Omitted from the section is the concept of equivalence of interest rates, all general annuities, and finding the interest rate for an annuity. Installment buying is treated without annuity symbols, which leads to an unnecessarily complicated method of determining the interest rate. The financial tables used have an unusually attractive format.

This book would not be adequate for the above-average student. It would be a pity for future leaders of business and industry to obtain the view that mathematics is so mechanical and lacking in concepts at a time when recent advances in mathematical theory and the use of electronic computing equipment holds such promise for the future.

C. L. SEEBECK, JR.
University of Alabama

*Integral Transforms in Mathematical Physics.* By C. J. Tranter. Wiley, New York, 1956. 133 pp. $2.00.

This is another fine little book in Methuen's Monographs on Physical Subjects. The emphasis is on the use of integral transforms in partial differential equations with chapters also on evaluation of integrals, combined use of relaxation methods and transforms, and a new chapter in this second edition on dual integral equations of the type arising from physical problems with one set of "mixed" boundary conditions.

The first four chapters deal with Laplace, Fourier, Hankel, and Mellin transforms, with their inversion formulae, and with a number of applications. The sixth chapter contains a discussion of finite transforms with applications of sine, cosine, Hankel, and Legendre transforms.

A fairly strong background in analysis is required for appreciation of the book as this analysis is naturally not presented in such a small book. A course in complex variables should suffice. Some knowledge of boundary value problems in mathematical physics is also a necessary prerequisite.

The examples or exercises are adequate for enhancing the understanding. The book should serve as a valuable supplement in or as a reference book for one who wishes to see whether he can apply integral transform methods to some particular problem.

R. B. DEAL
Oklahoma Agricultural and
Mechanical College

*Physics and Mathematics, Series I, Volume I. Progress in Nuclear Energy.* Edited by R. A. Charpie, et al. McGraw-Hill, New York, 1956. x+398 pp. $12.00.

This volume is an outgrowth of the United Nations Conference on the Peaceful Uses of Atomic Energy held at Geneva in 1955. It is devoted primarily to summarizing in great detail such information concerning neutron physics and fissionable nuclei as is needed for the design of nuclear reactors. Most of this data is here published for the first time, the internationally practised policy of withholding information on this subject for fifteen years having been relaxed specifically on the occasion of this conference. In their attempt to bridge this gap in the literature the authors of the eleven chapters in this book are in the anomalous position of correlating and distilling a literature which has been inaccessible to most readers. There are numerous descriptions of apparatus, empirical curves and tables. Yet the volume is not self contained. There is no introductory chapter to define the problems and indicate in what way the chapters in this book are addressed to them. Although numerous formulae occur in the first 250 pages, they are given only for comparison with empirical data. Neither their derivation nor the principles on which they are based are to be found here. Only in the ensuing 100 pages, in the chapters entitled "The Physics of Fast Reactors" and "Heterogeneous Methods for Calculating Reactors" does one find some attention to mathematics. In the former, a survey is given of mathematical methods used in calculating neutron transport phenomena. The equations considered in various approximations are the Boltzmann equation and the diffusion equation. The latter chapter, by S. M. Feinberg, of the U.S.S.R., seems to stand alone as a contribution of the Russian group in that no indication of related work done elsewhere is given. The exposition is correspondingly more self-contained and is of interest in that the theory is explicitly formulated for reactors with a 3-dimensional lattice structure.

The book appears to be of most value to designers of nuclear reactors. For others, it is most likely to be of interest only insofar as the text is a guide to the bibliography.

D. L. FALKOFF
Brandeis University

*Fundamental Concepts of Algebra.* By Claude Chevalley. Pure and Applied Mathematics Series, volume VII, Academic Press, New York, 1956. viii +241 pp. $6.80.

*Fundamental Concepts* offers a pronounced Bourbaki flavor. Indeed in a sense, the book may be considered as a welcome abridgment into the English language from Bourbaki's multi-volumed *Éléments de Mathématique.* The author, making no pretense to cover all the basic ideas in algebra, omits such topics as the theory of fields in order to elucidate modules and exterior algebras. He devotes approximately three-quarters of the pages in the volume to the third and fifth chapters, entitled respectively "Rings and Modules" and "Associative Algebras." The other three chapters discuss monoids, groups, and (very briefly) algebras.

The exposition is very closely knit, leaving the reader little opportunity to skip. The logical development advances steadily with a dynamic quality, so the reader also has little desire to skip. Yet the reader is not pampered. He must understand and assimilate the concepts quickly, lest he be unprepared to comprehend the subsequent sections. Within a section the arrangement of material is typically as follows: introductory description of new ideas, theorem, proof, theorem, proof, more new concepts, theorem, proof. Paragraphs are frequently very lengthy. Long proofs require high concentration by the reader; rarely is there found a hint of the future path for the argument or a summary of the portion of the assertions already proved. An excerpt from the preface concerning motivation is highly indicative of the style of writing: "what the student may learn here is not designed to help him with problems he has already met but with those he will have to cope with in the future; it is therefore impossible to motivate the definitions and theorems by applications of which the reader does not know the existence as yet." With this philosophy of presentation, the exposition displays considerable austerity. Examples are cited when certain algebraic structures are introduced, but illustrations for various significant theorems or major results are not attempted. The theory is relentlessly pushed onward.

A highlight of the book is the exercise list concluding each chapter. The problems, of which there are more than a hundred, seem unusually well chosen to offer the solver useful information and a broadening outlook on the theory.

There is a variety of errata which can easily be removed in a next printing and which, for the most part, will delay the reader only momentarily.

The preface suggests use of the text in a first graduate course. The desirability of every serious graduate student's learning the book's content at an early stage should not be challenged. Nevertheless the teacher of a beginning graduate student should warn him that studying this volume demands intensive work. The dividends paid will be well worth the effort, many times over.

R. A. Good
University of Maryland

## BRIEF MENTION

Publications of potential interest to mathematicians, but which are more properly reviewed in other periodicals, are described below.

*Mathematics for Electronics with Applications*. By Henry M. Nodelman and Frederick W. Smith. McGraw-Hill, 1956. vii+391 pp. $7.00.

This interesting book is one which is apt to be overlooked by mathematicians. It is quite possible that the misuse of technical mathematical vocabulary will so mitigate against this book that the important modern applications of mathematics to electronics will be lost. Statements such as, "a rectangular matrix is *equivalent to* a square matrix" (page 156) and "the determinant of a rectangular matrix is always zero" (page 159) and the implication on page 215 that the word "isomorphic" is synonymous with the phrase "one-to-one correspondence" will justifiably raise the ire of mathematicians, and make this book unsuitable as a mathematical text. Nevertheless, competent mathematicians may well wish to examine it for the many current applications which it contains. It is a sad commentary that the same publishing house which carefully prepared *Modern Mathematics for the Engineer* by Beckenbach would fail to correct the misuse of technical mathematical vocabulary in the volume here under discussion. This text could well have helped bridge the current gap between electrical engineering and mathematics, were it not for the misuse of certain important mathematical words.

One may shudder at the inclusion of the "cross-hatch" method of evaluating three by three determinants or wonder why, in today's world, more work on actual circuit design using Boolean algebra is not included. Still the collection of "up-to-date problems based on current engineering practice" should be welcomed by teachers seeking to teach mathematics to engineering students.

*Brief Analytic Geometry.* By Thomas E. Mason and Clifton T. Hazard. Ginn, 1957. 229 pp. $3.50.

The authors state that "no major changes in subject matter or methods of presentation have been made" in this third edition of *Brief Analytic Geometry.* "Changes have been made in the numerical data of many of the exercises. Several new exercises have been added. Few of the illustrative examples have been changed." This well-known book needs no additional review other than to mention a new edition has been prepared.

*Calculus Refresher for Technical Men.* By A. A. Klaf. Dover 1956. 431 pp. $1.95.

This paperback calculus refresher is in no way comparable to the excellent paperbacks by Oakley or by Graesser and Petersen.

*Trigonometry Refresher for Technical Men.* By A. A. Klaf. Dover Publications Inc., 1956. 629 pp. $1.95.

*The Icosahedron and the Solutions of Equations of the Fifth Degree.* By Felix Klein. Dover, 1956. xvi+289 pp. $1.85.

American geometers will indeed welcome the inexpensive paperback reprint of the English translation of Klein's historical volume on the Icosahedron, originally published in 1884 and translated into English in 1888. This work was reviewed extensively on pages 45–61, Vol. 9 (1887) *American Journal of Mathematics* by F. N. Cole.

*Table of the Fresnel Integral To Six Decimal Places.* By T. Pearcey. Cambridge University Press. 63 pp. $2.50.

These six and seven figure tables of Fresnel integrals will be welcomed by persons interested in diffraction theory. The clear type and adequate margins are a relief.

*Intermediate Algebra.* By Paul K. Rees and Fred W. Sparks. McGraw-Hill, 1956. x+306 pp. $3.90.

The authors state that the features of the first edition have been "all preserved" and that "the chief purpose in the preparation of the second edition of

*Intermediate Algebra* was to provide a selection of problems greater in number and more carefully graded than those in the first edition." Many teachers will undoubtedly welcome the new edition of this old favorite.

*Mathematics Magic and Mystery*. By Martin Gardner. Dover, 1956. xii+176 pp. $1.00.

This collection of mathematical and near mathematical tricks, puzzles and games is a welcome low cost addition to the library of anyone interested in such pastimes, and who isn't?

*Electronic Computers*. Edited by T. E. Ivall. Philosophical Library, 1956. 163 pp. $10.00.

*Electronic Computers* is not an advanced text for experts, but instead is definitely for the non-expert. Still it contains much of interest to mathematicians who are trembling on the brink of modern machine computation, both analogue and digital. While most of the machines mentioned are English, the American counterparts are well-known and the circuit designs and principles are international. A thoroughly enjoyable book giving general principles rather than specific "cook-book" directions.

## NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.*

### CONFERENCE ON MATRIX COMPUTATIONS

A Conference on Matrix Computations will be held at Wayne State University on September 3–6, 1957. The purpose of this conference is to bring together those persons who are concerned with the mathematical methods used in computing centers and who can communicate both in the technical language of digital computers and in the symbolism of matrix algebra.

Morning sessions will be devoted largely to invited addresses. Methods now being used to solve systems of linear equations, to compute the inverse of a matrix and to find characteristic values and characteristic vectors will be described. Papers suggesting new methods for the solution of standard problems will be solicited and an especial effort will be made to bring to attention new problems demanding the efforts of mathematicians. Smaller groups with well defined common interests will form discussion panels in the afternoon. It is expected that a report of methods in use at some of the main European centers will be given.

There is no tuition fee for the conference. Individuals who wish to present papers or to suggest speakers should contact Professor Wallace Givens, Chairman, Department of Mathematics, Wayne State University, Detroit 2, Michigan.

## SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1957.

*Catholic University of America*, June 26 to August 9: Dr. Ramler, college geometry, analytic projective geometry, ordinary differential equations; Dr. Moller, higher algebra I; Dr. Wiegmann, higher algebra II, introduction to matrix theory; Dr. Taam, advanced calculus I; Dr. Saworotnow, advanced calculus II; Dr. Finan, basic concepts of mathematics.

*Columbia University*, July 3 to August 16: Dr. Taft, introduction to higher algebra; Dr. Mendelson, differential equations; Mr. Gordon, probability; Professor Chevalley, fundamental concepts of mathematics, higher algebra; Professor Taylor, theory of functions of a real variable; Professor Feldman, theory of functions.

*Syracuse University*, July 1 to August 9: Professor Gelbart, analysis and applications I (differential equations); Professor Davis, an intermediate course in algebra, teaching high school mathematics; Professor Hemmingsen, history of mathematics; Professor Exner, analysis of elementary mathematics; Professor Gilchrist, programming for digital computers. August 12 to September 13: Professor Kostenbauder, analysis and applications II (vector analysis).

*University of California, Berkeley, Department of Statistics*, June 17 to July 27 and July 29 to September 7: Professor Neyman, individual research; Professors Neyman, Fix and Smith (University of Cambridge, England), research seminar in statistical problems of health. This course will be given in cooperation with Drs. Brooke, Hall, Serfling, and Willis (Communicable Disease Center, Public Health Service, Atlanta) and Dr. Mantel (National Institutes of Health, Bethesda). They will present the medico-biological side of practical problems preceding the statistical discussions.

*University of California, Los Angeles*, June 24 to August 2: Professor Horn, functions of a complex variable; Professor Straus, theory of relativity; Professor Bell, fundamental mathematical concepts.

*University of Colorado*, June 17 to August 23: Dr. Zirakzadeh, foundations of geometry; Professor McKelvey, topology; Professor Bunt, teaching of secondary school mathematics, mathematics workshop in teaching problems; Professor Magnus, history of mathematics, foundations of analysis; Mr. Householder, mathematical statistics; Professor Rogers, finite mathematics.

*University of Wisconsin*, July 1 to August 24: Professor F. B. Jones (University of North Carolina), topics in topology, advanced calculus; Dr. Walker (American Optical Co.), differential geometry; Dr. Artzy (Israel Institute of Technology), advanced topics in algebra, theory of probability; Professor Fadell, elementary topology, higher analysis; Professor Wagner, determinants and matrices; Dr. Kruskal, differential equations; Dr. Payne, foundations of algebra; Mr. Evey, theory and operation of computing machines.

## PERSONAL ITEMS

Dr. F. E. Browder, Yale University, has been awarded a National Science Foundation Postdoctoral Fellowship.

Dr. C. E. Shannon, a professor at Massachusetts Institute of Technology and a mathematical consultant in Bell Telephone Laboratories research department, has received the 1956 Research Corporation Award for his work in information theory.

*New Jersey State Teachers College, Montclair:* Associate Professor B. E. Meserve has been appointed Chairman of the Department of Mathematics; Dr. D. R. Davis, Chairman of the Department, has retired.

*University of Minnesota, College of Science, Literature and Arts:* Visiting Associate Professor Bjarni Jónsson, University of California, Berkeley, has been appointed Associate Professor; Associate Professor M. D. Donsker has been promoted to Professor; Assistant Professor W. S. Loud has been promoted to Associate Professor; Dr. G. E. Baxter and Dr. J. M. Slye have been promoted to Assistant Professors; Assistant Professor Ella Thorp has retired with the title Assistant Professor Emeritus.

*Wayne State University:* Dr. Karl Zeller, Tübingen University, Germany, has been appointed Associate Professor; Professor G. G. Lorentz is on leave of absence and has been appointed Visiting Professor at the University of Michigan. The School of Business Administration and the Computation Laboratory announce a new Master's Program in Automatic Data Processing.

Professor D. B. Ames, University of New Hampshire, has accepted a position as research mathematician with Hughes Aircraft Company, Culver City, California.

Dr. R. W. Bagley, Associate Research Scientist, Lockheed Aircraft Corporation, Sunnyvale, California, is on leave for the year to work on an operations research project at Stanford University.

Mr. H. W. Becker has been elected Secretary-Treasurer of the Omaha-Lincoln Section, Institute of Radio Engineers.

Assistant Professor Kurt Bing, Rensselaer Polytechnic Institute, has been promoted to Associate Professor.

Assistant Professor W. E. Briggs, University of Colorado, has been appointed Director of the University Academic Year Institute for Secondary School Teachers of Science and Mathematics sponsored by the National Science Foundation.

Professor Arthur Erdélyi, California Institute of Technology, is on leave and has been appointed Visiting Professor at Hebrew University, Jerusalem, Israel.

The annual award of the Duodecimal Society of America for 1956 has been given to Jean Essig, Inspector General of Finances for France.

Dr. F. G. Fisher, Navy Electronics Laboratory, San Diego, California, has accepted a position as a consulting mathematician with the U. S. Navy, Bureau of Ordnance, Washington, D. C.

Mr. J. L. Freier, is now a mathematician with Project Cyclone, Reeve Instrument Company, New York, New York.

Mr. R. M. Gordon has been appointed Supervisor, Customer Education, ElectroData Division, Burroughs Corporation, Pasadena, California.

Assistant Professor R. P. Gosselin, University of Connecticut, has been awarded a grant from the National Science Foundation for research in Fourier series.

Mr. F. D. Grogan, Quality Surety Office, Rocky Mountain Arsenal, Denver, Colorado, has a position as a systems analyst, Flight Controls Group, Glenn L. Martin Company, Denver.

Assistant Professor W. T. Guy, Jr., University of Texas, has been promoted to Associate Professor.

Mr. H. N. Hadley, Naval Powder Factory, Indian Head, Maryland, has accepted a position as senior reliability analyst with the AVCO Manufacturing Company, Lawrence, Massachusetts.

Mr. J. L. Hatfield, Mary Washington College, has been appointed Assistant Professor at College of William and Mary in Norfolk.

Associate Professor L. S. Hill, Hunter College, has been promoted to Professor.

Associate Professor R. C. James, Haverford College, has been appointed Professor and Chairman of the Department of Mathematics, Harvey Mudd College, effective September, 1957.

Mr. B. V. Lachapelle, Cornell University, has been appointed a research associate at the University of Montreal.

Professor Harry Langman, Detroit Institute of Technology, has been appointed Professor at Ohio Northern University.

Mr. J. G. Leghorn, University of Colorado, has accepted a position as an engineer with the Glenn L. Martin Company, Denver, Colorado.

Dean A. E. Meder, Jr., Rutgers University, is on leave of absence and has been appointed Executive Director of the Commission on Mathematics.

Mr. J. W. Mettler, Teacher, Trenton Central High School, New Jersey, has been appointed Assistant Professor at Pennsylvania State University.

Mr. George Millman, Analytical Statistician, Office of the Quartermaster General, Army Department, Washington, D. C., is now a mathematician at the Evans Signal Laboratories, Army Signal Corps, Ft. Monmouth, New Jersey.

Dr. M. E. Muller, Senior Mathematician, Scientific Computing Center, International Business Machines Corporation, New York City, is on leave of absence as a research associate at Princeton University.

Dr. R. Z. Norman, Princeton University, has been appointed Assistant Professor at Dartmouth College.

Mr. S. E. Puckette, Yale University, has been appointed Assistant Professor at the University of the South.

Professor Emeritus L. L. Silverman, Dartmouth College, has been appointed Visiting Professor at the University of Houston.

Dr. G. H. Swift, Duke University, has accepted a position as applied science representative with International Business Machines Corporation, Seattle, Washington.

Dr. E. D. Watters, Jr., Senior Mathematician, Bendix Research Laboratories, Detroit, Michigan, is an engineer at Westinghouse Electric Corporation, Baltimore, Maryland.

Dr. E. S. Wolk, University of Connecticut, has been promoted to Assistant Professor.

Mr. J. T. Yamada, University of Toronto, has been appointed Lecturer at McGill University.

Mr. Elmer Latshaw, a mechanical engineer, Naval Air Materiel Center, Philadelphia, Pennsylvania, died on January 18, 1957. He was a member of the Association for thirty-seven years.

Mr. J. M. Pellegrino, Mathematician, Electric Boat, Groton, Connecticut, died on March 19, 1956.

Miss Audrey I. Richards, Utica College of Syracuse University, died on September 15, 1956.

Professor A. C. Schaeffer, Chairman of the Department of Mathematics, University of Wisconsin, died on February 2, 1957. He was a member of the Association for nine years.

Brigadier General R. H. Somers died on January 22, 1957. He was a charter member of the Association.

Professor John von Neumann, Institute for Advanced Study, died on February 8, 1957. He was a member of the Association for twenty-four years.

# THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

## THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The nineteenth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of California, Berkeley, January 12, 1957. Professor H. L. Alder, Chairman of the Section, presided at the morning and afternoon general sessions. Two concurrent sessions were also held in the afternoon—one on the teaching of mathematics and one on research. Professor Alder presided at the former and Professor Harley Flanders, Vice-Chairman of the Section, at the latter. There were 156 persons in attendance at the meeting including 87 members of the Association.

Following Professor Blakeslee's report on the 1956 high school contest, the section voted unanimously to endorse the proposal that the Association sponsor a nationwide high school mathematics contest.

At the business meeting the following officers were elected for the coming year: Chairman, Professor Harley Flanders, University of California, Berkeley; Vice-Chairman, Professor B. J. Lockhart, U. S. Naval Postgraduate School; Secretary-Treasurer, Professor Roy Dubisch, Fresno State College.

By invitation of the section, Professor David Blackwell, University of California, Berkeley, delivered an address at the morning session entitled *Statistical Prediction of Sequences*. Abstract of this address follows:

A method of prediction of successive elements in an infinite sequence of zeros and ones, based on observation of previous elements, is described. The method has the property that, applied to any sequence, the proportion of correct predictions for the first $n$ elements of the sequence will be at least max $(p_n, 1-p_n)-\epsilon$ for all sufficiently large $n$, where $p_n$ is the proportion of ones in the first $n$ elements. An extension to more general statistical decision problems is indicated.

The following papers were presented:

1. *An approximation to the equally tempered musical scale*, by Professor I. J. Schoenberg, University of Pennsylvania and Stanford University.

The author presented some of the interesting historical researches of Professor J. M. Barbour on the equally tempered scale, contained in Barbour's article *A geometric approximation of the roots of numbers*, this MONTHLY, vol. 64, 1957, pp. 1–10.

2. *The Mathematical Association of America high school contest*, by Professor D. W. Blakeslee, San Francisco State College.

The procedures and results of the 1956 Annual Contest for high school students sponsored by the Northern California Section were reviewed. It was felt that the contest was highly successful; seventy-three schools and over 2300 students having participated.

3. *The visiting-lectureship program for high schools*, by Professor H. L. Alder, University of California, Davis.

Announcement is made of a program sponsored by the Northern California Section of the Mathematical Association of America whereby mathematicians from seven universities and colleges in Northern California are available to give lectures in high schools on a topic of general interest in mathematics at either a mathematics class, a special meeting arranged for the purpose, a school assembly, or (if a fairly good attendance can be assured) a mathematics club meeting. Invitation to avail themselves of this opportunity will be sent initially to about 70 high schools.

4. *Reflecting chess bishops*, by Professor S. Stein, University of California, Davis.

A reflecting chess bishop is a chess bishop which moves along a diagonal until it hits the border of the chess-board and then reflects off like a ray of light (if it hits a corner it reflects directly back). THEOREM: *Two reflecting bishops can be placed on an $m \times n$ chess-board to cover all squares if and only if* $(m-1, n-1)=1$.

5. *On Picture-Writing*, by Professor G. Polya, Stanford University.

This paper appeared in this MONTHLY, vol. 63, 1956, pp. 689–698.

6. *Factors of Fermat numbers*, by Professor R. M. Robinson, University of California, Berkeley.

Fermat believed that the numbers $F_m = 2^{2^m}+1$ are all prime, but Euler showed in 1732 that $F_5$ has the factor 641. Actually, no Fermat prime has ever been identified except $F_0=3$, $F_1=5$, $F_2=17$, $F_3=257$, and $F_4=65537$. The number $F_m$ is now known to be composite in twenty-nine cases, namely for $m=5$, 6, 7, 8, 9, 10, 11, 12, 15, 16, 18, 23, 36, 38, 39, 55, 63, 73, 117, 125, 144, 150, 207, 226, 228, 268, 284, 316, 452. In fourteen of these cases ($m>38$, $m \neq 73$), the compositeness was first established in 1956, using a high-speed computer (SWAC) and a program coded by the author. For example, $F_{207}$ was found to have the prime factor $3 \cdot 2^{209}+1$.

7. *An eigenvalue problem for ordinary differential equations*, by Professor S. P. Diliberto, University of California, Berkeley.

Let $W(t, \theta_2)$ be the matrix solution of the real linear (matrix) ordinary differential equation

$$(*) \qquad dW/dt = A(t, t+\theta_2)W$$

determined by the initial condition $W(0, \theta_2)=I$, where $A$ and $W$ are $n$-square, and $A(\theta_1, \theta_2)$ is continuous in $\theta_1, \theta_2$ with period $\omega_i$ in $\theta_i (i=1, 2)$.

Let $S$ denote the space of all continuous $\omega_2$-periodic $n$-vector functions of $\theta_2$, *e.g.*, $\alpha(\theta_2) \in S$ implies $\alpha(\theta_2)=(\alpha_1(\theta_2), \cdots, \alpha_n(\theta_2))$, where each $\alpha_i(\theta_2)$ is $C'$ and has period $\omega_2$ in $\theta_2$. A transformation $T: S \to S$ is defined by: $\gamma(\theta_2) \in S$, $(T\gamma)(\theta_2) \in S$, where $(T\gamma)(\theta_2)=W(\omega_1, \theta_2-\omega_1)\gamma(\theta_2-\omega_1)$. The eigenvalues of $T$ are studied and shown to characterize the limit $(t \to +\infty)$ behavior of the solutions of (*).

8. *A note on Rouche's theorem*, by Professor C. L. Clark, Oregon State College.

The usual condition of Rouche's theorem that $|g(z)| < |f(z)|$ on a simple closed curve $C$ lying in a simply connected region within which $g(z)$ and $f(z)$ are analytic can be replaced by the condition that $f(z)$ and $f(z)+g(z)$ lie in the same component of the mapping space $(Z-O)^C$, *i.e.*, the space of all continuous mappings of $C$ into $Z-O$, $Z$ being the complex plane and $O$ the origin. This generalization of Rouche's theorem is obtained by elementary use of the index function as developed by G. T. Whyburn and C. Kuratowski and is also an improvement of other generalizations. The results suggest further questions concerning zeros of functions.

9. *The teaching of interpolation*, by Professor H. A. Arnold, University of California, Davis.

Even in elementary courses it is important to teach the retention of extra "guard figures" in numbers used in numerical calculations. These numbers include input data and the results of interpolations. Teachable numerical examples are given to show this is feasible.

10. *A heuristic outlook in checking*, by Professor C. M. Larsen, San Jose State College.

Most teachers have observed students who, when asked to check their work, make wrong answers come out "right." Such students may be conditioned to force the checks when teachers say, for example, "To *check* a solution, put it into the given equation and *make sure* the equation is *satisfied.*" An alternative attitude, aimed at discovering error, rather than checking correctness, may be encouraged by advising, "To *test* a solution, put it into the given equation and *hunt for discrepancies.*" Some evidence was presented indicating that such advice may lead students to think more critically and constructively about their work.

11. *Some observations on teaching mathematics for the computer age*, by Professor Irving Sussman, University of Santa Clara.

A digest is made of diverse competent opinions on the impact which the emergence of electronic digital computers will have, or should have, on the teaching of undergraduate and preparatory mathematics. Although the expert opinions vary in details, there is general agreement that important changes both in course content and teaching methods have become necessary—this even in the pure mathematics curricula. The question of how such redirection of emphasis is to be brought about in view of the loss of such a large percentage of potential teaching personnel to industry, and in the face of traditional academic inertia, is posed as an unsolved problem.

ROY DUBISCH, *Secretary*

## CALENDAR OF FUTURE MEETINGS

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 31, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, May 4, 1957.
ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.
INDIANA, May 4, 1957.
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Johns Hopkins University, Baltimore, Maryland, May 4, 1957.
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Carleton College, Northfield, May 11, 1957.
MISSOURI
NEBRASKA
NEW JERSEY, Fall, 1957.
NORTHEASTERN, Dartmouth College, Hanover, New Hampshire, November 30, 1957.
NORTHERN CALIFORNIA, January 18, 1958.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, State College of Washington, Pullman, June 14, 1957.
PHILADELPHIA, November 28, 1957.
ROCKY MOUNTAIN, Colorado School of Mines, Golden, May 3–4, 1957.
SOUTHEASTERN
SOUTHERN CALIFORNIA, San Diego State College, May 11, 1957.
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.
WISCONSIN, Wisconsin State College, Whitewater, May 11, 1957.

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